Computers & Industrial Engineering 137 (2019) 106042

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

An alternative design of the two-sided CUSUM chart for monitoring the mean when parameters are estimated

Mandla D. Diko[a,*], Subha Chakraborti[b], Ronald J.M.M. Does[a]

[a] Department of Operations Management, University of Amsterdam, Plantage Muidergracht 12, 1018 TV Amsterdam, the Netherlands
[b] Department of Information Systems, Statistics and Management Science, University of Alabama, Tuscaloosa, AL 35487, USA

ARTICLE INFO

Keywords:
Cumulative sum control chart
Modified Siegmund formula
Unconditional average run length
Conditional average run length
Standard deviation of the average run length
Exceedance probability criterion

ABSTRACT

In recent literature on control charts, the exceedance probability criterion has been introduced to provide a minimum in-control performance with a specified probability. In this paper we evaluate the two-sided Phase II CUSUM charts and its in-control conditional average run length ($CARL_{IN}$) distribution with respect to the exceedance probability criterion. Traditionally, the $CARL_{IN}$ distribution and its parameters has been calculated by Markov Chains and simulations. We present in this paper a generalization of the Siegmund formula to calculate the $CARL_{IN}$ distribution and its parameters. This closed form formula is easy and faster to apply compared to Markov Chains. Consequently, we use it to make sample size recommendations and to adjust the charting constants via the exceedance probability criterion. The adjustments are done without bootstrapping. Results show that, in order to prevent low $CARL_{IN}$ values, more Phase I data are required than has been recommended in the literature. Tables of the adjusted charting constants are provided to facilitate chart implementation. The adjusted constants significantly improve the in-control performance, at the marginal cost of a lower out-of-control performance. Balancing the trade-off between the in-control and out-of-control performance is illustrated with real data and tables of charting constants.

1. Introduction

Since the in-control conditional average runlength ($CARL_{IN}$) depends on the Phase I parameter estimates, it is a random variable with its own probability distribution. It has been shown (cf. Chakraborti, 2006; Saleh, Zwetsloot, Mahmoud, & Woodall, 2016) that these distributions have large variability and therefore the values of the $CARL_{IN}$ can be quite different from the nominally specified average run length (ARL_0). In this context, the expected value of the $CARL_{IN}$ has been used to evaluate and design Phase II control charts. This approach, the so-called unconditional perspective, has received a lot of attention in the literature. See, for example, Abbasi, Riaz, and Miller (2012), Chakraborti (2000), Diko, Chakraborti, and Graham (2016), Diko, Goedhart, Chakraborti, Does, and Epprecht (2017), Goedhart, Schoonhoven, and Does (2016), Jardim, Chakraborti, and Epprecht (2018), and Sanusi, Abujiya, Riaz, and Abbas (2017). However, the unconditional perspective does not show individual chart performance, which is known to vary from practitioner to practitioner.

Accordingly, another approach, the so-called conditional perspective, has been suggested. Under this perspective, a number of criteria has been used to answer different problems. To study the performance of the one-sided and two-sided CUSUM location charts, Jones, Champ, and Rigdon (2004) used the mean, the standard deviation and the 10th, 50th, and 90th percentiles of the conditional run length distribution. Jeske (2016) presented a modified Siegmund formula (Siegmund, 1985) for approximating the $CARL_{IN}$ of the upper one-sided CUSUM chart to derive Phase I sample size requirements to ensure probabilistic control of the relative error of the $CARL_{IN}$. Saleh et al. (2016) used the standard deviation of the $CARL_{IN}$ to quantify the amount of variation in the in-control CUSUM chart performance corresponding to different amounts of Phase I data. In recent literature on control charts (cf. Diko, Chakraborti, & Does, 2019; Epprecht, Loureiro, & Chakraborti, 2015), Phase I sample size requirements have been derived on the basis of the exceedance probability criterion (Albers & Kallenberg, 2004). In this paper, we use the exceedance probability criterion to derive the number of Phase I subgroups required to design a two-sided CUSUM control chart for the mean. To calculate the values of the $CARL_{IN}$ distribution, we extend the Jeske (2016) modified Siegmund formula from the one-sided CUSUM chart to the two-sided case. This is compared with the Markov Chain method, which is a popular method to calculate the $CARL_{IN}$ values. It will be shown that the extended modified Siegmund formula is accurate and easy to use. Our Phase I subgroup requirements turn out to be larger than those of Jeske (2016), Jones et al.

* Corresponding author.
E-mail address: m.d.diko@uva.nl (M.D. Diko).

https://doi.org/10.1016/j.cie.2019.106042
Received 30 November 2018; Received in revised form 31 August 2019; Accepted 3 September 2019
Available online 04 September 2019
0360-8352/

(2004) and Saleh et al. (2016). However, in practice, the available Phase I data can be limited. To compensate for limited Phase I data, we adjust the control limits according to the exceedance probability criterion.

This paper is structured as follows. In Section 2 we give an overview of the two-sided Phase II CUSUM control chart for monitoring the mean of a normal process based on some popular estimators for the unknown process mean and variance. In Section 3, we derive the modified Siegmund formula to calculate the $CARL_{IN}$ of the two-sided Phase II CUSUM control chart and compare it with the well-known Markov Chain method. In Section 4, we introduce the exceedance probability criterion to obtain the $CARL_{IN}$ prediction bounds. In Section 5, we find the minimum number of Phase I subgroups required to design a two-sided Phase II CUSUM control chart. Results are compared with the available results in the literature. In Section 6, the charting constants are adjusted without using the bootstrap method (cf. Gandy & Kvaloy, 2013; Jones & Steiner, 2012). Tables of the adjusted constants are given. A comparison between our method and the bootstrap method is made. In Section 7, the application of the adjusted and unadjusted control charts is illustrated using real life data from Montgomery (2013). The adjusted and unadjusted limits charts are compared in terms of their in-control and out-of-control performance. Possible trade-offs to balance the in-control and out-of-control performance of the adjusted limits charts are suggested. Finally, in Section 8, a summary and conclusions are offered.

2. Two-sided CUSUM charts with estimated parameters

Let X_{ij}, $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, denote the in-control Phase I data from a normal distribution with an unknown mean μ_0 and unknown standard deviation σ_0. To estimate μ_0, we use the estimator $\hat{\mu}_0 = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}\big/mn$. To estimate σ_0, for $n > 1$, we use the pooled standard deviation estimator

$$\hat{\sigma}_{01} = S_p\Big/c_4(m(n-1)+1) = \frac{1}{c_4(m(n-1)+1)}\sqrt{\frac{1}{m}\sum_{i=1}^{m} S_i^2} \tag{1}$$

where c_4 is the unbiasing constant for the estimator S_p assuming the normal distribution (Montgomery, 2013) and S_i is the standard deviation of the i^{th} Phase I sample. For $n > 1$, there are several other estimators of σ_0 that can be used. Popular among these are the mean range and the mean standard deviation estimators. However, the results we obtained using these estimators marginally differ from the results we obtained using $\hat{\sigma}_{01}$. Hence, we only focus on $\hat{\sigma}_{01}$. Furthermore, when the number of Phase I subgroups m is moderately large, the constant c_4 becomes indistinguishable from 1 (cf. Diko et al., 2017; Mahmoud, Henderson, Epprecht, & Woodall, 2010), hence, we may use $\hat{\sigma}_{01} = S_p$. For $n = 1$, we use the average moving ranges as the estimator for σ_0

$$\hat{\sigma}_{02} = MR\Big/d_2(2)(m-1) = \sum_{i=2}^{m}|X_i - X_{i-1}|\Big/1.128(m-1) \tag{2}$$

where $d_2(2)$ is an unbiasing constant (cf. Montgomery, 2013).

Assuming that the Phase II data are normally distributed with mean μ and standard deviation σ, for $i = m+1, m+2, \cdots$ the standardized charting statistics of the two-sided CUSUM control chart for the mean are given by

$$C_i^+ = max(0, C_{i-1}^+ + W_i - k) \quad \text{and} \quad C_i^- = min(0, C_{i-1}^- + W_i + k) \tag{3}$$

where

$$W_i = \frac{\bar{X}_i - \hat{\mu}_0}{\hat{\sigma}_{0g}/\sqrt{n}} = \left[\gamma T_i + \sqrt{n}\delta - \frac{Z}{\sqrt{m}}\right]Q_g^{-1},\ Q_g = \frac{\hat{\sigma}_{0g}}{\sigma_0} = a_g\sqrt{\frac{Y}{b_g}},\ \gamma = \frac{\sigma}{\sigma_0},\ \delta = \frac{\mu - \mu_0}{\sigma_0},$$

$T_i = \frac{\sqrt{n}(\bar{X}_i - \mu)}{\sigma}$, $Z = \frac{\sqrt{mn}(\hat{\mu}_0 - \mu_0)}{\sigma_0}$, $\bar{X}_i$ is the i^{th} sample mean, Y is a chi-square variable with b_g degrees of freedom. We use the subscript $g = 1, 2$ to distinguish between the two unbiased estimators of σ_0 that we employ in this paper. The constants b_g and a_g $(g = 1, 2)$ are functions of m and n. If $g = 1$ then $b_1 = m(n-1)$ and $a_1 = 1$. If $g = 2$ then we calculate b_2 and a_2 using the Patnaik (1950) approximation as described in Saleh et al. (2016). The calculation of b_2 and a_2 can be found in the R codes in Appendix A. We assume that the starting values (C_m^+ and C_m^-) are both equal to zero, which means that the process is initially in-control. Note that, selecting $C_m^+ > 0$ and/or $C_m^- < 0$ gives the chart a fast initial response or "head start", since it increases the charts ability to detect a process that is initially out-of-control (cf. Lucas & Crosier, 1982). We also assume that the reference values (k^+ and k^-) are both equal to k. The reference value k is usually chosen to be half the shift (δ) that is considered important enough to be detected.

The standardized two-sided Phase II CUSUM control chart gives an out-of-control signal when

$$C_i^+ \geq h^+ \quad \text{or} \quad C_i^- \leq -h^- \tag{4}$$

where $h^+ = h^- = h$ is the charting limit to be found. For the parameters known case, the h values can be found by specifying ARL_0 and k in the function xcusum.crit(k,ARL_0,$sided$ = "two") of the R package "spc". As an example, for some combinations of ARL_0 and k, the h values are given in Table 1.

Since the h values in Table 1 are for the parameters known case, they do not account for the effects of parameter estimation. Unless the number of Phase I subgroups m is very large, these charting constants should not be used to construct Phase II control charts. However, most practitioners continue to use them regardless of the value of m. This can be attributed to the fact that adjusting these charting constants for the effects of parameter estimation is computer intensive. Moreover, no tables, graphs or software packages for the adjusted h values are conveniently available to help practitioners to correctly implement their Phase II CUSUM control charts with estimated parameters. Hence, in this paper, the tables of adjusted constants are presented, so that the two-sided CUSUM chart for the mean with estimated parameters can be implemented more easily in practice.

3. The modified Siegmund formula for the $CARL_{IN}$ of the two-sided Phase II CUSUM control chart

Traditionally, the $CARL_{IN}$ values of CUSUM charts have been calculated by simulations and Markov Chains (cf. Brook & Evans, 1972; Woodall, 1984). Recently, Jeske (2016) presented a modified Siegmund formula for approximating the $CARL_{IN}$ of the upper one-sided CUSUM chart. We extend this formula to the two-sided CUSUM chart and compare the results with the results of the well-known Markov Chain method.

The original Siegmund approximation (Siegmund, 1985) for the in-control average run length (ARL_{IN}) of the upper or the lower one-sided CUSUM with reference value (k) and control limit (h), when the parameters are known, is given by

Table 1
Values of k and their corresponding values of h for the two-sided CUSUM control chart and ARL_0 = 100, 200, 370.

ARL_0	k	h
100	0.12	7.968
	0.25	5.597
	0.50	3.502
200	0.12	9.998
	0.25	6.854
	0.50	4.172
370	0.12	12.083
	0.25	8.008
	0.50	4.774

$$ARL_{IN}(k,h)=\frac{\exp(2k(h+1.166))-2k(h+1.166)-1}{2k^2}. \tag{5}$$

For the upper one-sided CUSUM chart, when parameters are estimated, Jeske (2016) showed that the signaling event $C_i^+ \geq h$ is equivalent to

$$max(0, C_{i-1}^+ + \gamma T_i + \sqrt{n}\delta - (U+kV)) \geq hV \tag{6}$$

where $U=\frac{Z}{\sqrt{m}}$, $V=Q_g=a_g\sqrt{\frac{Y}{b_g}}$ and in which one can identify the modified "reference value" as $U+kV$ and the modified "control limit" as hV, respectively. By substituting the modified "reference value" and the modified "control limit" into Eq. (6), Jeske (2016) proposed the modified Siegmund formula for approximating the $CARL_{IN}$ of the upper (C_i^+) one-sided CUSUM chart as

$$\begin{aligned}
&CARL_{IN}^+(Z,Y,k,h,m,n)\\
&=\frac{\exp(2(U+kV)(hV+1.166))-2(U+kV)(hV+1.166)-1}{2(U+kV)^2}\\
&=\frac{\exp\left[2\left(\frac{Z}{\sqrt{m}}+ka_g\sqrt{\frac{Y}{b_g}}\right)\left(ha_g\sqrt{\frac{Y}{b_g}}+1.166\right)\right]-2\left(\frac{Z}{\sqrt{m}}+ka_g\sqrt{\frac{Y}{b_g}}\right)\left(ha_g\sqrt{\frac{Y}{b_g}}+1.166\right)-1}{2\left(\frac{Z}{\sqrt{m}}+ka_g\sqrt{\frac{Y}{b_g}}\right)^2}.
\end{aligned} \tag{7}$$

Note that the $CARL_{IN}^+$ is a random variable, being a function of the two random variables Z and Y. Likewise, for the lower (C_i^-) one-sided CUSUM charting statistic one can rewrite the signaling event $C_i^- \leq -h$. Then simplify in a similar manner with the "reference value" $-U+kV=-\frac{Z}{\sqrt{m}}+ka_g\sqrt{\frac{Y}{b_g}}$ and the modified "control limit" $hV=ha_g\sqrt{\frac{Y}{b_g}}$, respectively, and express the modified Siegmund formula for approximating the $CARL_{IN}$ of the lower (C_i^-) one-sided CUSUM control chart as

$$\begin{aligned}
&CARL_{IN}^-(Z,Y,k,h,m,n)\\
&=\frac{\exp\left[2\left(-\frac{Z}{\sqrt{m}}+ka_g\sqrt{\frac{Y}{b_g}}\right)\left(ha_g\sqrt{\frac{Y}{b_g}}+1.166\right)\right]-2\left(-\frac{Z}{\sqrt{m}}+ka_g\sqrt{\frac{Y}{b_g}}\right)\left(ha_g\sqrt{\frac{Y}{b_g}}+1.166\right)-1}{2\left(-\frac{Z}{\sqrt{m}}+ka_g\sqrt{\frac{Y}{b_g}}\right)^2}.
\end{aligned} \tag{8}$$

Now, using the Van Dobben de Bruyn (1968) formulation for calculating the average runlength of the two-sided tabular CUSUM chart, the modified Siegmund formula for approximating the $CARL_{IN}$ of the two-sided CUSUM control chart can be expressed as

$$CARL_{IN}(Z,Y,k,h,m,n)=\left[\frac{1}{CARL_{IN}^+(Z,Y,k,h,m,n)}+\frac{1}{CARL_{IN}^-(Z,Y,k,h,m,n)}\right]^{-1}, \tag{9}$$

where $CARL_{IN}^+(Z,Y,k,h,m,n)$ and $CARL_{IN}^-(Z,Y,k,h,m,n)$ are given in Eqs. (7) and (8), respectively.

In Figs. 1 and 2, Q-Q plots are used to illustrate the quality of the modified Siegmund approximation (Eq. (9)) relative to the Markov Chain approximation for the $CARL_{IN}$ for $n = 1, 5$, $m = 1000$, $ARL_0 = 200$ and $k = 0.25, 0.50, 0.75, 1$. The annotated R codes for applying the modified Siegmund formula and Markov Chain approximation to calculate the empirical $CARL_{IN}$ distributions are given in Appendix A.

Looking at Figs. 1 and 2, it can be seen that for $k = 0.25, 0.50$ and 0.75, the modified Siegmund approximation based $CARL_{IN}(z,y,k,h,m,n)$ values are similar to their corresponding Markov Chain values. But, for $k = 1$, they are consistently higher than their Markov Chain counterparts. The latter is more pronounced when $n = 5$ than $n = 1$. Note that in practice we use $k \leq 0.75$, since smaller shifts are of more interest while using a CUSUM chart. Thus, there is no practical need for using the Markov Chain method to calculate the $CARL_{IN}$ of the two-sided Phase II CUSUM control chart. We recommend in these cases Eq. (9) because it is simple and more practical.

The modified Siegmund approximation formula for the $CARL_{IN}$ given in Eq. (9) is also convenient for approximating some important CUSUM control chart performance metrics. For example, the mean of the $CARL_{IN}$, denoted by $AARL_{IN}$,

$$\begin{aligned}
AARL_{IN}(k,h,m,n)&=E(CARL_{IN}(Z,Y,k,h,m,n))\\
&=\int_{-\infty}^{\infty}\int_0^{\infty}CARL_{IN}(z,y,k,h,m,n)\phi(z)q(y)dydz
\end{aligned} \tag{10}$$

and the standard deviation of the $CARL_{IN}$, denoted by $SDARL_{IN}$,

$$\begin{aligned}
&SDARL_{IN}(k,h,m,n)\\
&=\sqrt{Var(CARL_{IN}(Z,Y,k,h,m,n))}\\
&=\sqrt{E(CARL_{IN}(Z,Y,k,h,m,n))^2-(E(CARL_{IN}(Z,Y,k,h,m,n)))^2}\\
&=\left[\int_{-\infty}^{\infty}\int_0^{\infty}(CARL_{IN}(z,y,k,h,m,n))^2\phi(z)q(y)dydz-(AARL_{IN})^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{11}$$

Looking at Eqs. (10) and (11), note that $CARL_{IN}(z,y,k,h,m,n)$ is given in (9), q is the pdf of a chi-square distribution with b_g degrees of freedom ($g = 1, 2$) and ϕ is the pdf of the standard normal distribution. In Fig. 3, for a range of m values, we compare the Siegmund and Markov Chain (see Appendix A) approximation methods. Note that, the horizontal line at $CARL_{IN} = 200$ represents the nominal ARL_0, while the other two horizontal lines are drawn one nominal $SDARL_{IN}$ (10% * ARL_0) away from $ARL_0 = 200$.

From Fig. 3, for $k \leq 0.75$ and all m, it can be seen that the Siegmund and Markov Chain methods give similar results. It can also be seen that, for all k, m affects the $AARL_{IN}$ and the $SDARL_{IN}$. For example, for $k = 0.25$, for both the Siegmund and Markov Chain methods, increasing m to 500 has an impact of reducing the $SDARL_{IN}$ to within *10%* of the $ARL_0 = 200$. Furthermore, increasing m to *1000* increases the $AARL_{IN}$ to $ARL_0 = 200$. We conclude that the accuracy of the modified Siegmund formula is not dependent on m and that the modified Siegmund formula leads to the same conclusions that have been reached by others such as Saleh et al. (2016), who used the Markov Chain method. However, the modified Siegmund formula is much simpler and more convenient to use.

In the next section, we use the modified Siegmund formula to find the $CARL_{IN}$ prediction bounds based on the exceedance probability criterion.

4. Prediction bounds

Saleh et al. (2016) used the $SDARL_{IN}$ to study the effects of parameter estimation on the Phase II CUSUM control chart. The $SDARL_{IN}$ indicates the amount of spread of the $CARL_{IN}$ values around the $AARL_{IN}$. However, it does not distinguish between low $CARL_{IN}$'s and high $CARL_{IN}$'s. Thus, it cannot explicitly indicate how much of the $CARL_{IN}$ variability is due to the low $CARL_{IN}$'s and how much is due to the high $CARL_{IN}$'s, which is very important to know, because the goal is to minimize the low $CARL_{IN}$'s and maximize the high $CARL_{IN}$'s. On the other hand, the $CARL_{IN}$ prediction bounds can be used to provide boundaries to separate low $CARL_{IN}$*'s* from high $CARL_{IN}$*'s* values and to indicate the proportion of each. Hence, to study the effects of parameter estimation, we advocate using the exceedance probability criterion over the $SDARL_{IN}$.

The exceedance probability criterion involves setting up the prediction bounds (or one-sided prediction intervals). As in Epprecht et al. (2015), an upper one-sided prediction bound for the $CARL_{IN}$ can be defined as

$$P(CARL_{IN}(Z,Y,k,h,m,n) \geq ARL_0)=1-p, \tag{12}$$

where ARL_0 is the specified lower prediction bound and p is a probability. A small value of p such as *0.05* is desirable, since it means that there will be a small proportion (*5%*) of low $CARL_{IN}$'s ($CARL_{IN}$'s that are less than ARL_0) and hence a large proportion (*95%*) of $CARL_{IN}$'s greater than ARL_0 This is desirable from a practical point of view.

To apply the exceedance probability, the distribution of the $CARL_{IN}$ has to be found or at least approximated. For time-weighted charts, such as the EWMA and CUSUM charts, the exact distribution is not

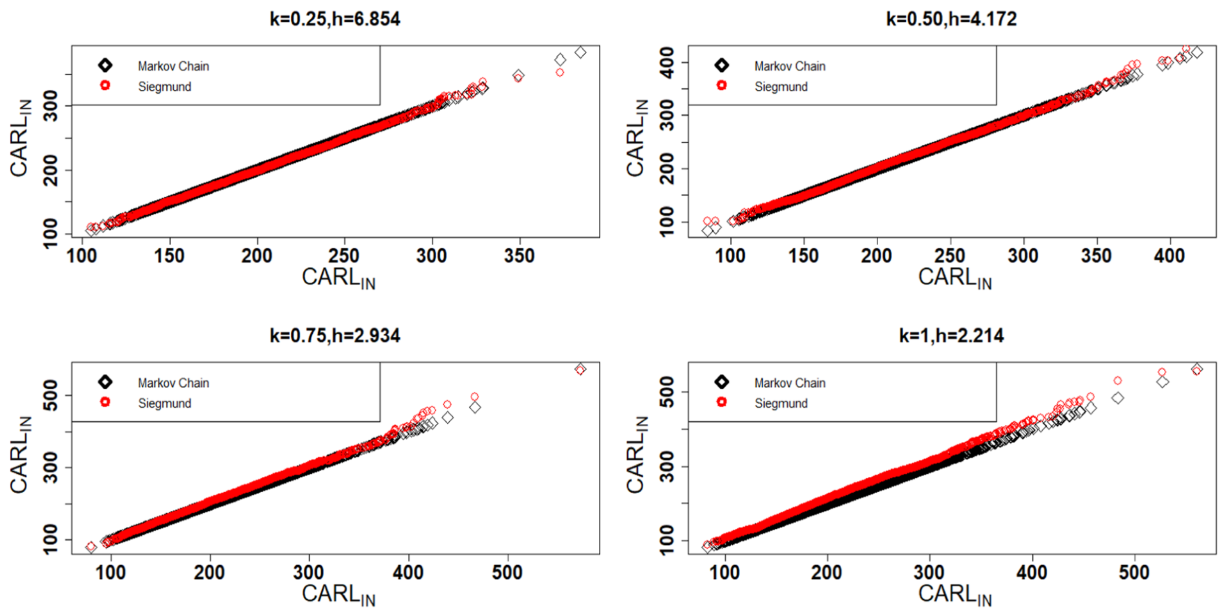

Fig. 1. QQ-plots for the empirical $CARL_{IN}$ distributions of the two sided CUSUM charts for location when $m = 1000$, $n = 1$ and $ARL_0 = 200$.

available. Diko et al. (2019) approximated the $CARL_{IN}$ distribution of the two-sided EWMA chart by an empirical distribution (F_N). Their F_N was obtained by generating many Phase I subgroups, calculating the corresponding $CARL_{IN}$ values by using the Markov Chain method and then ordering them in ascending order. We will adopt this approach. However, to calculate the $CARL_{IN}$ values, we use Eq. (9) instead of Markov Chains. Once F_N has been found, we calculate its $100p^{th}$ percentile, denoted by $CARL_{IN,p}$. The $CARL_{IN,p}$ is the observed (or the estimated) $CARL_{IN}$ upper prediction bound, which is compared with the nominal ARL_0. The ARL_0 is the theoretical value that must be exceeded with high probability *1-p*. The comparison between $CARL_{IN,p}$ and ARL_0 is based on the percentage difference

$$PD = \frac{CARL_{IN,p} - ARL_0}{ARL_0} \times 100. \tag{13}$$

The R code for generating F_N and the algorithm for finding *PD* are given in Appendix A and Appendix B, respectively, while the results are given in Table 2.

Table 2 shows the $CARL_{IN,p}$'s and *PD*'s of the standard Phase II CUSUM control chart for $k = 0.25, 0.50$ and $ARL_0 = 200$ for different combinations of *m, n* and *p*. From Table 2, it can be seen that, for all *n*, nearly all the $CARL_{IN,p}$'s are less than or equal to ARL_0. Hence, nearly all of the *PD*'s are negative. Thus, in addition to high $CARL_{IN}$'s, the observed $CARL_{IN}$ prediction bounds include low $CARL_{IN}$'s. For this reason, the Phase II CUSUM control charts, designed using the unadjusted *h* values (which are shown in Table 1), will have a lower than nominally expected $CARL_{IN}$, which means a deterioration in chart performance. It can also be seen that, for all *n* and small *m*, the $CARL_{IN,p}$ and *PD* are far less than ARL_0 and 0, respectively. Moreover, as *m* increases, $CARL_{IN,p}$ and *PD* converge to ARL_0 and 0, respectively.

Based on these results, we can already infer that it will take huge numbers of Phase I subgroups to make $CARL_{IN,p}$ close enough to ARL_0 or *PD* close to 0. In the next section, we investigate, in more detail, the question of how large *m* should be to design charts.

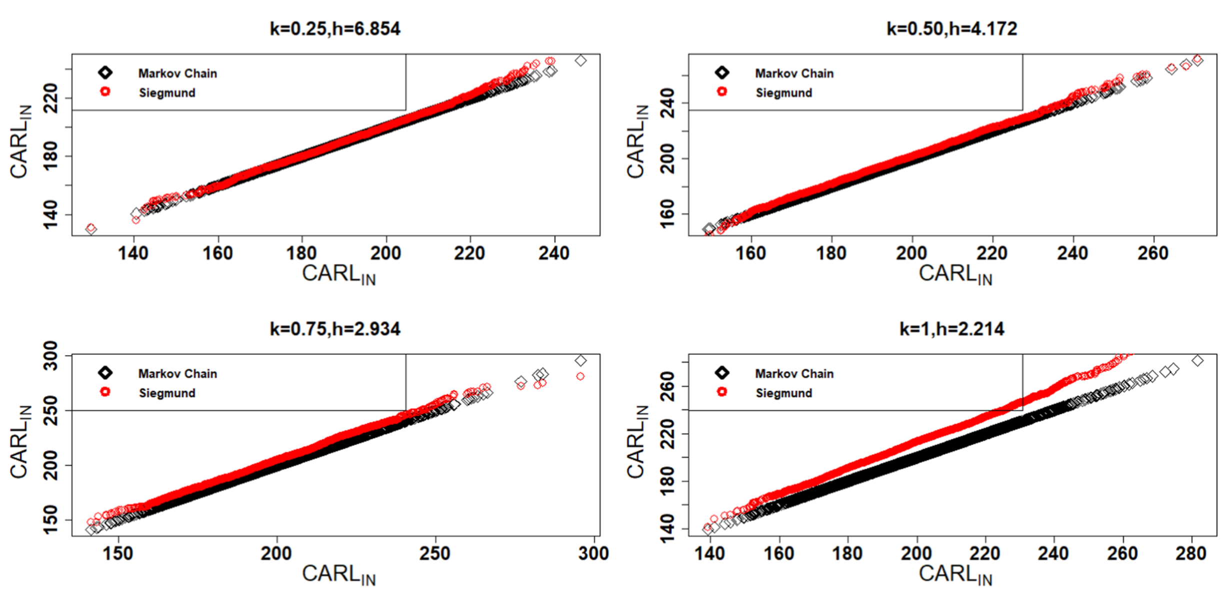

Fig. 2. QQ-plots for the empirical $CARL_{IN}$ distributions of the two sided CUSUM charts for location when $m = 1000$, $n = 5$ and $ARL_0 = 200$.

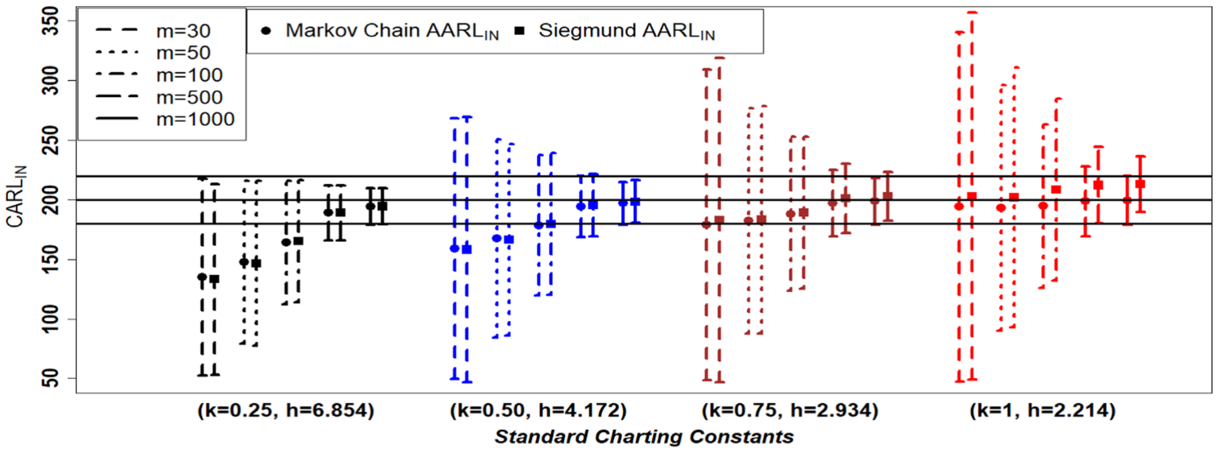

Fig. 3. $AARL_{IN}$ and $SDARL_{IN}$ for a two sided CUSUM chart for $n = 5$ and $ARL_0 = 200$.

5. Required number of Phase I samples

For the upper one-sided CUSUM control chart, Jeske (2016) determined the minimum m required to guarantee with $1-\alpha$ probability that the relative error (RE) will be different from 0 by at most ε

$$P(|RE| < \varepsilon) = 1-\alpha$$
$$P\left(-\varepsilon < \frac{CARL_{IN}(Z,Y,k,h,m,n)-ARL_0}{ARL_0} < \varepsilon\right) = 1-\alpha$$
$$P(ARL_0(1-\varepsilon) < CARL_{IN}(Z,Y,k,h,m,n) < ARL_0(1+\varepsilon)) = 1-\alpha. \tag{14}$$

This is not equal to the exceedance probability criterion. In this situation, the quantity α is the sum of the lower and higher tail probabilities of the distribution of $CARL_{IN}$. Thus, the m values that are recommended on the basis of Eq. (14) do not explicitly protect against high proportions of low $CARL_{IN}$'s.

In Saleh et al. (2016) the m recommendations were made to keep the $SDARL_{IN}$ down at, $ARL_0 \times \varepsilon$, where $\varepsilon = 0.1$. Note that, the $SDARL_{IN}$ is calculated from the deviations of $CARL_{IN}(Z,Y,k,h,m,n)$ from $AARL_{IN}$ and the $AARL_{IN}$ is not necessarily equal to ARL_0, unless m is very large. If m is large, then $CARL_{IN}(Z,Y,k,h,m,n) \approx Normal(ARL_0, SDARL_{IN})$, because of the central limit theorem. Consequently, since the area under the normal curve, within one standard deviation of the mean, is approximately 68%, it can be said that the Saleh et al. (2016) m recommendations were made to guarantee with *0.68* probability that the difference

Table 2
$CARL_{IN,p}$ and PD values of a two-sided CUSUM chart as a function of m, n for k = *0.25, 0.50; p = 0.05, 0.10;* and ARL_0 = 200.

			m									
p	n	k	30	50	200	500	750	1000	3000	5000	10,000	50,000
0.05	1	0.25	28.78	41.96	92.09	125.68	138.25	145.98	168.29	175.42	182.71	192.52
			−85.61	−79.02	−53.95	−37.16	−30.87	−27.01	−15.86	−12.29	−8.65	−3.74
		0.50	24.66	37.55	85.39	117.92	130.25	138.37	162.73	170.89	179.48	191.52
			−87.67	−81.23	−57.31	−41.04	−34.88	−30.81	−18.64	−14.56	−10.26	−4.24
	5	0.25	37.31	53.46	114.68	150.72	162.31	168.98	184.84	188.97	192.75	197.27
			−81.34	−73.27	−42.66	−24.64	−18.85	−15.51	−7.58	−5.52	−3.63	−1.37
		0.50	44.42	64.69	126.02	155.64	164.79	170.20	184.30	188.37	192.40	197.65
			−77.79	−67.66	−36.99	−22.18	−17.60	−14.91	−7.85	−5.81	−3.80	−1.17
	10	0.25	38.27	54.57	117.23	154.31	166.27	172.95	188.23	191.82	194.90	198.29
			−80.87	−72.72	−41.38	−22.85	−16.87	−13.53	−5.88	−4.09	−2.55	−0.86
		0.50	47.61	69.24	134.80	164.19	172.76	177.52	189.16	192.30	195.28	198.97
			−76.19	−65.38	−32.56	−17.91	−13.62	−11.24	−5.42	−3.85	−2.36	−0.52
	20	0.25	38.56	55.17	118.20	155.95	168.04	174.71	190.05	193.45	196.18	198.93
			−80.72	−72.42	−40.90	−22.03	−15.98	−12.65	−4.98	−3.27	−1.91	−0.54
		0.50	48.91	71.24	138.59	168.75	177.18	181.75	192.15	194.75	197.07	199.80
			−75.55	−64.38	−30.71	−15.62	−11.41	−9.13	−3.92	−2.62	−1.46	−0.10
0.10	1	0.25	36.67	52.24	105.45	136.96	148.29	155.27	174.38	180.35	186.34	194.21
			−81.67	−73.88	−47.28	−31.52	−25.86	−22.37	−12.81	−9.83	−6.83	−2.89
		0.50	33.04	48.52	100.05	130.91	142.13	149.55	170.18	177.04	184.07	193.69
			−83.48	−75.74	−49.97	−34.54	−28.93	−25.23	−14.91	−11.48	−7.97	−3.15
	5	0.25	47.51	66.73	129.25	160.37	169.71	175.04	187.89	191.21	194.33	197.97
			−76.25	−66.64	−35.38	−19.81	−15.15	−12.48	−6.06	−4.40	−2.84	−1.02
		0.50	57.04	79.47	138.28	163.70	171.39	175.88	187.72	191.08	194.34	198.52
			−71.48	−60.27	−30.86	−18.15	−14.31	−12.06	−6.14	−4.46	−2.83	−0.74
	10	0.25	48.95	68.39	132.59	164.58	173.89	179.00	190.70	193.56	196.04	198.76
			−75.53	−65.81	−33.71	−17.71	−13.06	−10.50	−4.65	−3.22	−1.98	−0.62
		0.50	61.72	85.82	147.17	171.30	178.26	182.09	191.64	194.21	196.61	199.57
			−69.14	−57.09	−26.42	−14.35	−10.87	−8.95	−4.18	−2.90	−1.70	−0.21
	20	0.25	49.31	69.21	133.96	166.34	175.87	180.98	192.38	194.93	197.07	199.25
			−75.34	−65.40	−33.02	−16.83	−12.07	−9.51	−3.81	−2.54	−1.46	−0.37
		0.50	63.61	88.59	151.83	175.76	182.29	185.81	194.04	196.13	198.02	200.21
			−68.19	−55.71	−24.09	−12.12	−8.85	−7.10	−2.98	−1.93	−0.99	0.11

Table 3
Four Phase II CUSUM control charts for $ARL_0 = 200$. Each constructed using the parameters known case limits with m values that satisfy $ARL_0 \times \varepsilon$, where $\varepsilon = 0.1$.

	$(n = 1, m = 3000, k = 0.25, h = 6.854)$	$(n = 1, m = 5000, k = 0.50, h = 4.172)$
$AARL_{IN}$	199.3	202
$SDARL_{IN}$	20.2	20.3
$P(CARL_{IN} < AARL_{IN}(1-\varepsilon))$	0.16	0.16
$P(CARL_{IN} < ARL_0(1-\varepsilon))$	0.17	0.13
	$(n = 5, m = 600, k = 0.25, h = 6.854)$	$(n = 5, m = 800, k = 0.50, h = 4.172)$
$AARL_{IN}$	190.9	197.6
$SDARL_{IN}$	20.8	20.2
$P(CARL_{IN} < AARL_{IN}(1-\varepsilon))$	0.16	0.16
$P(CARL_{IN} < ARL_0(1-\varepsilon))$	0.29	0.19

between $CARL_{IN}(Z, Y, k, h, m, n)$ and $AARL_{IN} = ARL_0$ will be no more than $ARL_0 \times \varepsilon$. This makes the $SDARL_{IN}$ criteria a special case of Eq. (14) with $1 - \alpha = 0.68$, which implies a lower (and an upper) tail probability of approximately $p = \alpha/2 = 0.16$. To illustrate this in more detail, consider the following chart parameters ($n = 1$, $k = 0.25$, $h = 6.854$), ($n = 1$, $k = 0.50$, $h = 4.172$), ($n = 5$, $k = 0.25$, $h = 6.854$) and ($n = 5$, $k = 0.50$, $h = 4.172$). For these chart parameters, Saleh et al. (2016) recommended $m = 3000$, $m = 5000$, $m = 600$ and $m = 800$, respectively. Using these recommendations, we evaluated these four CUSUM charts according to the $AARL_{IN}$, $SDARL_{IN}$, $P(CARL_{IN} < AARL_{IN}(1-\varepsilon))$ and $P(CARL_{IN} < ARL_0(1-\varepsilon))$. The results are shown in Table 3.

From Table 3, it can be seen that in terms of the $SDARL_{IN}$ the four CUSUM charts are performing as nominally specified. Except for the ($n = 5$, $m = 600$, $k = 0.25$, $h = 6.854$) chart, the $AARL_{IN}$'s are very close to $ARL_0 = 200$. For all charts, it can be seen that $P(CARL_{IN} < AARL_{IN}(1-\varepsilon)) = 0.16$, so the central limit theorem applies. Moreover, since $AARL_{IN} \cong ARL_0$, it can also be seen that $P(CARL_{IN} < ARL_0(1-\varepsilon)) \cong 0.16$, which confirms our explanation. Therefore, even though p is not explicitly mentioned, Saleh et al. (2016)'s m values do in fact reduce the frequent occurrence of low $CARL_{IN}$'s to 16%. But, if a smaller ε is used, the probability of an unsatisfactory $CARL_{IN}$ will be much higher than what might be deemed acceptable. For example, for the ($n = 1$, $m = 5000$, $k = 0.50$, $h = 4.172$) if $\varepsilon = 0.05$ then p = 37%. This suggests that in practice, one should better consider the exceedance probability criterion instead of the $SDARL_{IN}$ as a chart performance and design criteria.

Given ε, most practitioners would be interested in higher levels of protection, such as $p = 0.05, 0.10$, (so that the exceedance probability in (12) is high, such as *90%* or *95%*) and would naturally want to know about the minimum m required to achieve these levels. In technical terms, the practitioner may want to know the minimum m required to guarantee with probability $1-p$ that the RE will be less than 0 by at most ε, where p is an explicitly specified small proportion, such as *0.05 or 0.10*. To answer this, a method that explicitly controls the lower tail area, p, of the $CARL_{IN}$ distribution is needed. This can be found by simply considering only the lower prediction bound (and not the interval) of Eq. (14). Thus,

$$P(RE > -\varepsilon) = 1 - p$$

$$P\left(\frac{CARL_{IN}(Z, Y, k, h, m, n) - ARL_0}{ARL_0} > -\varepsilon\right) = 1 - p$$

so that $$P(CARL_{IN}(Z, Y, k, h, m, n) > ARL_0(1-\varepsilon)) = 1 - p. \quad (15)$$

This is the exceedance probability criterion that was used by Diko et al. (2019), Epprecht et al. (2015) and Loureiro, Epprecht, Chakraborti, and Jardim (2018) to recommend Phase I subgroup

Table 4
Required minimum number of Phase I subgroups, m, as a function of ε, p, k, n and ARL_0 = 100, 200, 370.

			ε = 10%		ε = 20%		ε = 30%	
ARL_0	n	k	$p = 0.05$	$p = 0.10$	$p = 0.05$	$p = 0.10$	$p = 0.05$	$p = 0.10$
100	1	0.25	5300	3350	1300	900	550	400
		0.50	7550	4700	1850	1200	800	500
	5	0.25	1300	900	450	300	250	200
		0.50	1440	1000	400	300	250	200
	10	0.25	1050	700	350	260	200	150
		0.50	850	600	300	233	200	150
	20	0.25	900	650	350	260	200	150
		0.50	650	450	250	180	200	150
200	1	0.25	7600	4800	1850	1250	800	550
		0.50	10,500	6500	2600	1650	1100	700
	5	0.25	1950	1400	700	500	350	250
		0.50	2000	1350	600	450	300	200
	10	0.25	1500	1100	600	450	350	250
		0.50	1200	850	450	300	250	150
	20	0.25	1300	950	550	400	350	250
		0.50	900	650	350	250	200	150
370	1	0.25	10,650	6850	2650	1800	1150	800
		0.50	14,050	8750	3450	2200	1450	950
	5	0.25	2900	2050	1050	750	550	400
		0.50	2700	1800	800	600	400	300
	10	0.25	2250	1600	950	650	550	400
		0.50	1650	1150	600	450	350	250
	20	0.25	2050	1450	900	600	500	350
		0.50	1200	900	500	350	300	200

Table 5
Adjusted charting constants when $n = 5$ and $\varepsilon = 0$.

		$ARL_0 = 100$				$ARL_0 = 200$				$ARL_0 = 370$			
p	k	m = 30	m = 50	m = 100	m = 200	m = 30	m = 50	m = 100	m = 200	m = 30	m = 50	m = 100	m = 200
0.05	0.12	24.99	17.91	12.29	9.81	49.12	33.72	20.50	14.42	90.16	60.68	33.76	20.79
	0.14	23.30	16.31	11.22	9.12	45.36	30.21	18.08	13.01	82.94	53.82	28.32	18.02
	0.16	21.52	14.89	10.37	8.49	41.59	26.88	15.99	11.79	75.80	47.00	23.90	15.89
	0.18	19.79	13.61	9.51	7.99	37.96	23.65	14.25	10.85	68.69	40.48	20.50	14.20
	0.20	18.15	12.41	8.86	7.46	34.28	20.77	12.87	9.97	61.58	34.40	17.78	12.86
	0.22	16.64	11.35	8.25	7.05	30.60	18.35	11.65	9.32	54.74	28.94	15.70	11.74
	0.24	15.15	10.45	7.76	6.67	27.35	16.27	10.67	8.67	47.84	24.47	14.09	10.84
	0.26	13.79	9.67	7.29	6.31	24.20	14.55	9.89	8.12	41.34	20.93	12.74	10.04
	0.28	12.63	8.97	6.87	5.97	21.30	13.11	9.20	7.70	35.28	18.15	11.65	9.35
	0.30	11.53	8.34	6.45	5.75	18.82	11.82	8.58	7.22	29.88	16.00	10.70	8.79
	0.32	10.63	7.81	6.15	5.45	16.70	10.86	8.05	6.86	25.17	14.34	9.97	8.28
	0.34	9.84	7.34	5.86	5.25	14.87	10.02	7.52	6.52	21.47	12.98	9.29	7.79
	0.36	9.07	6.89	5.59	4.99	13.38	9.35	7.16	6.20	18.64	11.88	8.72	7.42
	0.38	8.51	6.53	5.32	4.83	12.19	8.69	6.79	5.98	16.46	10.93	8.17	7.07
	0.40	7.95	6.17	5.07	4.65	11.10	8.16	6.40	5.70	14.72	10.14	7.74	6.73
	0.42	7.41	5.83	4.92	4.43	10.28	7.68	6.12	5.49	13.30	9.43	7.33	6.43
	0.44	6.98	5.60	4.70	4.30	9.55	7.25	5.85	5.25	12.17	8.83	7.01	6.13
	0.46	6.65	5.35	4.55	4.17	8.90	6.87	5.59	5.07	11.22	8.35	6.65	5.93
	0.48	6.23	5.13	4.35	4.03	8.35	6.54	5.35	4.85	10.37	7.87	6.37	5.67
	0.50	5.92	4.92	4.22	3.91	7.81	6.21	5.20	4.71	9.71	7.49	6.11	5.48
	0.52	5.70	4.72	4.03	3.73	7.43	5.96	4.98	4.56	9.06	7.12	5.86	5.26
	0.54	5.41	4.53	3.93	3.63	7.02	5.65	4.75	4.35	8.51	6.79	5.61	5.10
	0.56	5.21	4.34	3.82	3.54	6.64	5.44	4.64	4.24	8.07	6.48	5.38	4.88
	0.58	4.93	4.23	3.64	3.44	6.33	5.23	4.50	4.13	7.65	6.15	5.24	4.75
	0.60	4.74	4.06	3.56	3.35	6.02	5.02	4.32	4.01	7.30	5.92	5.02	4.62
0.10	0.12	19.91	14.51	10.81	9.21	38.12	25.92	16.79	13.01	69.25	45.27	25.58	17.75
	0.14	18.25	13.30	9.92	8.62	34.51	22.91	14.91	11.81	61.92	38.93	21.72	15.71
	0.16	16.66	12.09	9.19	8.07	30.90	20.16	13.39	10.87	55.10	32.90	18.80	14.06
	0.18	15.21	11.11	8.61	7.58	27.45	17.74	12.15	10.04	48.20	27.77	16.50	12.70
	0.20	13.86	10.23	8.05	7.15	24.27	15.77	11.07	9.27	41.58	23.43	14.68	11.68
	0.22	12.65	9.45	7.55	6.75	21.45	14.05	10.24	8.74	35.51	20.13	13.22	10.74
	0.24	11.57	8.77	7.07	6.37	18.89	12.73	9.47	8.17	29.97	17.47	12.06	9.97
	0.26	10.61	8.21	6.71	6.11	16.71	11.60	8.82	7.72	25.30	15.53	11.04	9.34
	0.28	9.84	7.67	6.37	5.77	14.81	10.61	8.29	7.31	21.66	13.93	10.25	8.75
	0.30	9.13	7.22	6.05	5.55	13.41	9.82	7.79	6.92	18.68	12.60	9.50	8.28
	0.32	8.45	6.75	5.75	5.32	12.16	9.14	7.36	6.56	16.48	11.58	8.96	7.78
	0.34	7.94	6.44	5.46	5.06	11.12	8.52	6.92	6.29	14.69	10.69	8.39	7.39
	0.36	7.46	6.09	5.28	4.89	10.28	8.00	6.60	6.00	13.31	9.91	7.92	7.02
	0.38	7.02	5.83	5.03	4.72	9.49	7.49	6.29	5.76	12.07	9.26	7.53	6.76
	0.40	6.57	5.56	4.87	4.47	8.88	7.10	6.00	5.50	11.14	8.64	7.14	6.44
	0.42	6.23	5.31	4.63	4.33	8.31	6.72	5.72	5.32	10.33	8.13	6.83	6.19
	0.44	5.98	5.09	4.50	4.20	7.83	6.44	5.53	5.05	9.63	7.73	6.52	5.93
	0.46	5.67	4.87	4.34	4.07	7.37	6.09	5.29	4.89	9.05	7.35	6.25	5.72
	0.48	5.45	4.65	4.15	3.94	6.95	5.85	5.05	4.75	8.47	6.97	5.97	5.47
	0.50	5.21	4.52	4.04	3.82	6.64	5.61	4.91	4.60	8.01	6.68	5.71	5.31
	0.52	4.98	4.33	3.91	3.63	6.28	5.38	4.75	4.38	7.62	6.36	5.54	5.13
	0.54	4.73	4.20	3.73	3.53	6.04	5.15	4.55	4.25	7.21	6.11	5.31	4.91
	0.56	4.61	4.04	3.64	3.44	5.74	4.94	4.44	4.14	6.87	5.88	5.15	4.78
	0.58	4.43	3.93	3.54	3.34	5.53	4.81	4.30	4.03	6.55	5.65	4.95	4.64
	0.60	4.26	3.76	3.46	3.26	5.32	4.62	4.12	3.92	6.31	5.42	4.81	4.50

numbers for Phase II Shewhart and EWMA control charts. Thus, given k, h, n, p, ε, and ARL_0 values, Eq. (15) can be solved for m. The algorithm for solving m is given in Appendix C and the results are shown in Table 4.

Table 4 reveals that, for a given ARL_0, m decreases with an increase of n, ε and p. For example, for $n = 1$, $ARL_0 = 200$, $k = 0.25$, $\varepsilon = 0.10$, m varies from about *7600* (when $p = 0.05$) down to *4800* (when $p = 0.10$). Thus, it is seen that to keep the probability of low $CARL_{IN}$'s less than 0.16, more Phase I data are required than has been recommended in Saleh et al. (2016). Where big data is available, we encourage the practitioners to use Table 4 along with the charting constants from the R package "spc". Where data are scarce, the option is to adjust h as a function of the available m.

6. Adjusting the CUSUM limits

In this section, we adjust the control limits of the two-sided Phase II CUSUM control chart according to the exceedance probability criterion. To apply the exceedance probability criterion to adjust the control limits, the bootstrap method (cf. Gandy & Kvaloy, 2013; Jones & Steiner, 2012) is used. It is known that repeated application of the bootstrap method would most likely result in different solutions (cf. Saleh et al., 2016). To overcome this limitation, Hu and Castagliola (2017) advise to run over again the bootstrap method a 100 times and averaging the results. For Shewhart charts, assuming a normal process, Goedhart, da Silva, et al. (2017), Goedhart, Schoonhoven, and Does (2017, 2018) and Jardim, Chakraborti, and Epprecht (2018) provided closed form analytical expressions for the adjusted constants based on the exceedance probability criterion. Hence in this situation bootstrapping was not necessary. Moreover, Diko et al. (2019) gave an alternative method for finding the adjusted constants. For comparison and completeness, we present our method (Diko et al., 2019) and the bootstrap method in Appendix D and E, respectively.

From Appendices D and E it can be seen that our method differs

Table 6
Adjusted charting constants when $n = 5$ and $\varepsilon = 20\%$.

		$ARL_0 = 100$				$ARL_0 = 200$				$ARL_0 = 370$			
p	k	m = 30	m = 50	m = 100	m = 200	m = 30	m = 50	m = 100	m = 200	m = 30	m = 50	m = 100	m = 200
0.05	*0.12*	20.29	14.74	10.52	8.75	39.58	27.49	17.33	12.79	72.39	48.98	28.01	18.21
	0.14	18.88	13.59	9.74	8.17	36.48	24.73	15.45	11.63	66.69	43.61	23.96	16.03
	0.16	17.50	12.48	9.05	7.67	33.52	22.13	13.88	10.68	60.98	38.17	20.63	14.29
	0.18	16.19	11.50	8.44	7.22	30.70	19.76	12.55	9.85	55.41	33.16	17.96	12.92
	0.20	14.94	10.59	7.88	6.81	27.82	17.55	11.41	9.17	49.77	28.58	15.86	11.78
	0.22	13.70	9.80	7.40	6.45	25.02	15.71	10.49	8.54	44.41	24.40	14.17	10.82
	0.24	12.69	9.09	6.97	6.12	22.48	14.11	9.68	8.02	38.96	21.05	12.77	10.02
	0.26	11.69	8.48	6.58	5.83	20.07	12.74	8.99	7.54	34.01	18.32	11.65	9.34
	0.28	10.77	7.91	6.23	5.55	17.95	11.60	8.38	7.13	29.25	16.17	10.72	8.74
	0.30	9.95	7.41	5.92	5.31	15.97	10.65	7.87	6.75	25.13	14.42	9.93	8.22
	0.32	9.26	6.98	5.64	5.08	14.41	9.82	7.41	6.42	21.60	13.05	9.25	7.76
	0.34	8.62	6.59	5.39	4.88	13.05	9.12	7.01	6.13	18.79	11.88	8.67	7.35
	0.36	8.05	6.24	5.15	4.68	11.87	8.52	6.64	5.85	16.55	10.92	8.15	6.99
	0.38	7.53	5.93	4.94	4.51	10.85	7.97	6.31	5.60	14.80	10.12	7.71	6.67
	0.40	7.10	5.64	4.74	4.34	10.05	7.50	6.01	5.37	13.36	9.41	7.29	6.37
	0.42	6.69	5.38	4.54	4.18	9.34	7.08	5.75	5.16	12.17	8.81	6.91	6.09
	0.44	6.33	5.13	4.38	4.04	8.67	6.71	5.50	4.96	11.17	8.30	6.60	5.85
	0.46	6.03	4.92	4.22	3.90	8.15	6.38	5.28	4.79	10.34	7.81	6.30	5.61
	0.48	5.71	4.71	4.07	3.77	7.66	6.07	5.07	4.61	9.64	7.42	6.04	5.41
	0.50	5.46	4.52	3.93	3.65	7.23	5.79	4.87	4.46	9.00	7.02	5.78	5.20
	0.52	5.21	4.36	3.81	3.55	6.84	5.54	4.70	4.31	8.48	6.70	5.56	5.02
	0.54	4.99	4.19	3.67	3.43	6.51	5.32	4.54	4.17	7.98	6.41	5.35	4.86
	0.56	4.79	4.05	3.57	3.33	6.20	5.10	4.38	4.04	7.57	6.11	5.15	4.69
	0.58	4.59	3.91	3.46	3.24	5.91	4.90	4.23	3.92	7.18	5.86	4.97	4.54
	0.60	4.42	3.78	3.35	3.14	5.65	4.72	4.09	3.79	6.86	5.62	4.79	4.40
0.10	*0.12*	16.22	12.21	9.45	8.32	30.81	21.51	14.52	11.65	55.77	36.89	21.92	15.88
	0.14	14.94	11.24	8.78	7.79	27.99	19.15	13.07	10.70	49.99	31.96	19.00	14.18
	0.16	13.81	10.37	8.21	7.33	25.20	17.03	11.90	9.90	44.59	27.38	16.63	12.83
	0.18	12.70	9.62	7.70	6.92	22.61	15.23	10.89	9.19	39.24	23.50	14.79	11.71
	0.20	11.67	8.94	7.23	6.55	20.22	13.72	10.04	8.59	34.19	20.28	13.28	10.79
	0.22	10.77	8.33	6.83	6.21	18.01	12.41	9.30	8.07	29.42	17.67	12.08	10.00
	0.24	9.96	7.80	6.47	5.91	16.09	11.32	8.67	7.60	25.30	15.65	11.09	9.33
	0.26	9.24	7.31	6.13	5.64	14.44	10.41	8.12	7.17	21.70	13.99	10.24	8.75
	0.28	8.61	6.90	5.83	5.38	13.03	9.60	7.64	6.81	18.90	12.66	9.52	8.23
	0.30	8.04	6.50	5.56	5.15	11.86	8.91	7.20	6.47	16.63	11.57	8.89	7.78
	0.32	7.55	6.17	5.32	4.94	10.85	8.33	6.82	6.17	14.76	10.65	8.36	7.37
	0.34	7.10	5.87	5.09	4.75	10.01	7.84	6.48	5.90	13.32	9.86	7.89	7.02
	0.36	6.70	5.58	4.88	4.56	9.29	7.36	6.18	5.65	12.16	9.20	7.46	6.69
	0.38	6.33	5.33	4.69	4.39	8.67	6.96	5.90	5.42	11.16	8.64	7.10	6.40
	0.40	6.00	5.11	4.52	4.24	8.11	6.60	5.64	5.21	10.31	8.13	6.75	6.13
	0.42	5.71	4.88	4.34	4.08	7.64	6.28	5.41	5.01	9.59	7.68	6.43	5.87
	0.44	5.45	4.70	4.19	3.95	7.19	5.98	5.19	4.82	8.98	7.27	6.16	5.65
	0.46	5.21	4.50	4.04	3.81	6.82	5.73	5.00	4.66	8.45	6.90	5.90	5.43
	0.48	4.99	4.33	3.90	3.69	6.47	5.48	4.81	4.49	7.95	6.59	5.67	5.24
	0.50	4.78	4.18	3.78	3.57	6.17	5.26	4.64	4.34	7.52	6.29	5.45	5.05
	0.52	4.60	4.05	3.66	3.47	5.89	5.05	4.48	4.20	7.15	6.02	5.25	4.88
	0.54	4.42	3.90	3.54	3.36	5.64	4.87	4.33	4.07	6.80	5.78	5.07	4.72
	0.56	4.26	3.78	3.44	3.26	5.40	4.69	4.19	3.94	6.49	5.55	4.89	4.57
	0.58	4.11	3.66	3.34	3.17	5.18	4.52	4.06	3.82	6.20	5.34	4.73	4.42
	0.60	3.97	3.54	3.24	3.08	4.97	4.36	3.93	3.71	5.95	5.15	4.58	4.29

from the bootstrap method in two significant ways:

1. In Appendix D, the search interval for h has a lower bound and the solution (say h^*) is very close to this lower bound. In Appendix E, the search interval is unbounded and so to find h^* is more demanding.
2. In Appendix D, h^* is found such that $CARL_{IN,p} = ARL_0(1-\varepsilon)$ is satisfied, that is, the solution emanates directly from the definition $P(CARL_{IN} > ARL_0(1-\varepsilon)) = 1-p$. However, in Appendix E, h^* is the $(1-p)^{th}$ percentile of the in-control distribution of the values of h, which satisfy $CARL_{IN} = ARL_0(1-\varepsilon)$. Although it works, it is not based on the definition of the exceedance probability criterion, which is why practitioners find the bootstrap method difficult.

Since our method is more logical, less arduous and more precise than parametric bootstrapping, we used it to find the adjusted charting limits.

Tables 5 and 6 presents the adjusted limits for $n = 5$, $m = 30, 50, 100, 200$, $0.12 \le k \le 0.60$, $ARL_0 = 100, 200, 370$, $\varepsilon = 0, 0.20$, and $p = 0.05, 0.10$. Since Tables 5 and 6 do not cover all the values, an R code to get some of the uncovered values is given in Appendix F. The use of Tables 5, 6 and Appendix F is outlined below.

Suppose we want to detect a $\delta = 1$ shift in the process mean by using a two sided CUSUM chart for location. Note that $\delta = \frac{\mu - \mu_0}{\sigma_0}$. What h^* should we use to guarantee that at most $p = 10\%$ of the $CARL_{IN}$'s are not more than $ARL_0 = 200$? Assume that $m = 30$ and $n = 5$. Since $\delta = 1$, then $k = \delta/2 = 0.50$. Since ε is not mentioned, we should use Table 5 ($\varepsilon = 0$). Looking at Table 5 at $p = 0.10$, $ARL_0 = 200$, $k = 0.50$, $m = 30$, $n = 5$, it can be seen that $h^* =$

Table 7
Hard bake process data for Phase II and the calculations for the standardized *CUSUM* charting statistics.

Phase II Sample #	$\bar{X}$	W	k	C^-	C^+
31	1,472	−1,24707	0,5	−1,84094	0
32	1,5292	1,012808	0,5	−1,32813	1,512808
33	1,5317	1,112275	0,5	−0,71585	3,125083
34	1,57934	3,007709	0,5	0	6,632792
35	1,4279	−3,01758	0,5	−3,51758	4,115216
36	1,48238	−0,85	0,5	−4,86758	3,765214
37	1,49098	−0,50784	0,5	−5,87541	3,757378
38	1,61278	4,338173	0,5	−2,03724	8,595551
39	1,65598	6,056955	0,5	0	15,15251
40	1,64202	5,501534	0,5	0	21,15404
41	1,67156	6,676831	0,5	0	28,33087
42	1,62516	4,830732	0,5	0	33,6616
43	1,69696	7,687411	0,5	0	41,84901
44	1,63214	5,108442	0,5	0	47,45746
45	1,77	10,59343	0,5	0	58,55088

6.64. Further, suppose we use another δ, whose corresponding k is not in Table 5. For example $\delta = 0.98$ ($k = 0.49$). For this case the h^* solution is between $h^* = 6.64$ *(k = 0.50)* and $h^* = 6.99$ *(k = 0.48)*. To find it, we employ a search algorithm, written in R and given in Appendix F. Before running this code, the lower bound of the search interval has to be correctly specified. In this case the lower bound is *6.64 (k = 0.50)*. Running the code gives $h^* = 6.82$, which is the adjusted constant we should use when $k = 0.49$.

If we want to improve the out-of-control performance of the chart, we can increase ε. Table 6 shows some h^* values when $\varepsilon = 20\%$. Looking at Table 6 at $p = 0.10$, $ARL_0 = 200$, $k = 0.50$, $m = 30$, it can be seen that $h^* = 6.17$. If ε is not in the Tables, we can use the same logic as in the above paragraph. For example, if $\varepsilon = 10\%$, it can be seen from Tables 5 and 6 that the h^* solution is between $h^* = 6.17$ ($\varepsilon = 20\%$) and $h^* = 6.94$ ($\varepsilon = 0$), respectively. To find it, we use Appendix F with $\varepsilon = 10\%$, $p = 0.10$, $ARL_0 = 200$, $k = 0.50$, $m = 30$ and *6.17* as the lower bound of the search interval (h). Running this R code shows that $h^* = 6.42$.

To improve the out-of-control performance, we can also increase p. From Table 5 at $ARL_0 = 200$, $k = 0.50$, $m = 30$, $n = 5$, it can be seen that if $p = 0.05$ then $h^* = 7.81$. However, for $p = 0.07$, the h^* solution is between $h^* = 6.64$ *(p = 10%)* and $h^* = 7.81$ *(p = 5%)*. To find it, we use the R code in Appendix F with $p = 0.07$, $ARL_0 = 200$, $k = 0.50$, $m = 30$, $n = 5$ and *6.64* as the lower bound of the search interval (h). Running this R code shows that $h^* = 7.24$.

For $n = 1$, the appropriate h^* values can also be found by running the R code in Appendix F. In this case the lower bound of the search interval should be set equal to the corresponding h^* value of the $n = 5$ case. For example, to find h^* for $\varepsilon = 0$, $p = 0.10$, $ARL_0 = 200$, $k = 0.50$, $m = 30$, $n = 1$, we would use Appendix F with $\varepsilon = 0$, $p = 0.10$, $ARL_0 = 200$, $k = 0.50$, $m = 30$, $n = 1$ and *6.64* ($n = 5$ or Table 5) as the lower bound of the search interval. Doing this gives $h^* = 8.31$. This is the adjusted constant we should use when $n = 1$.

For m values that are not covered by the Tables, we can also use Appendix F to find the corresponding h*. For example, if $m = 25$, we can run the R code in Appendix F with $\varepsilon = 0$, $p = 0.10$, $ARL_0 = 200$, $k = 0.50$, $m = 25$, $n = 5$ as parameters and specify the lower bound as *6.64 (m = 30)*. Running this code gives $h^* = 7.20$.

Next, real data from Montgomery (2013) is used to illustrate the implementation of the two sided CUSUM chart for the mean.

7. Implementing the two sided CUSUM chart for the mean: An example using real data

For our illustration, we consider the hard bake process data in Montgomery (2013). The data consist of 45 samples of size $n = 5$. We use the first $m = 30$ in-control samples as our Phase I data to calculate the mean and standard deviation estimates. We found the Phase I mean and standard deviation estimates to be *1.504* and *0.138*, respectively. Table 7 shows the sample means of the remaining 15 Phase II samples. Table 7 also uses these samples to calculate the standardized Phase II charting statistics. Fig. 4 plots these standardized Phase II charting statistics on the standardized two-sided Phase II CUSUM control chart.

From Fig. 4, it can be seen that the chart has two sets of control limits, the adjusted limits ($h^* = 6.64$) and the unadjusted limits ($h = 4.172$). These limits were chosen to detect a shift of size $\delta = 1$ in the process mean. The unadjusted limits are taken from Table 1 using $ARL_0 = 200$, $k = 0.50$. The adjusted limits are taken from Table 5 using $p = 0.10$, $ARL_0 = 200$, $k = 0.50$, $m = 30$, $n = 5$. From Fig. 4, it can also be seen that the adjusted limits are wider than the unadjusted limits. This has implications on the performance of the chart.

Figs. 5 and 6 show the boxplots of the in-control and out-of-control *CARL* distribution, respectively, of the limits in Fig. 4. From Figs. 5 and 6, it can be seen that the unadjusted control limits (narrower limits)

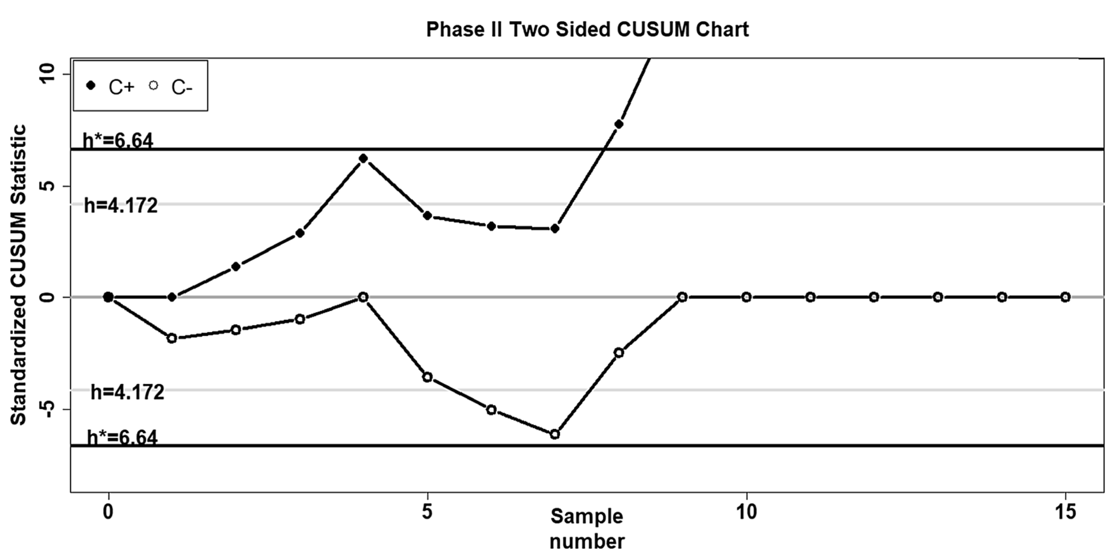

Fig. 4. A Phase II two side CUSUM Control chart to monitor the mean of the hard bake process data with adjusted (h^*) and unadjusted limits (h) when *m = 30, n = 5*, *ARL₀ = 200* and *p = 0.10*.

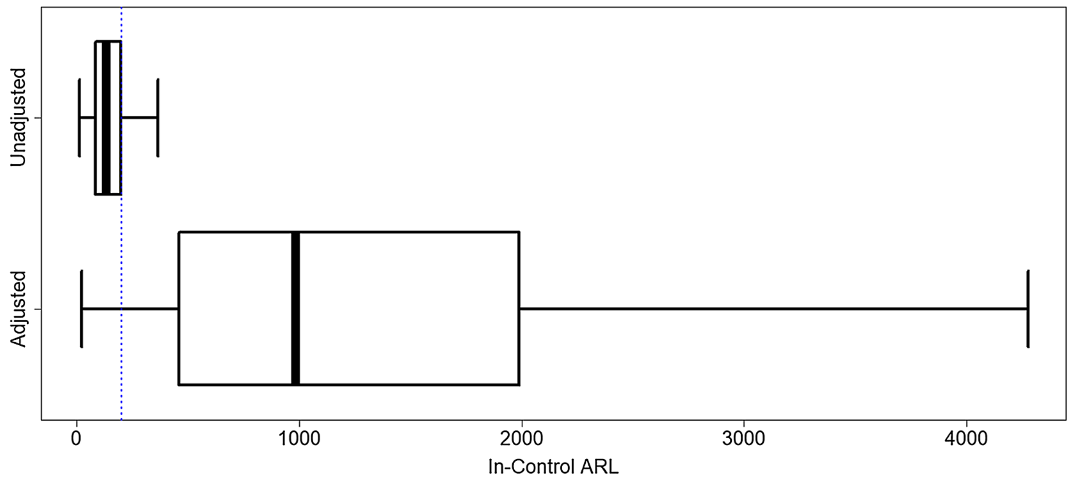

Fig. 5. Boxplots of the $CARL_{IN}$ distribution of the two sided Phase II CUSUM chart for the mean of the hard bake process data.

have a poor in-control performance and a slightly better out-of-control performance compared to the adjusted limits (wider limits).

It can be desirable to improve the out-of-control performance of the adjusted limits. This means using narrower adjusted control limits (smaller adjusted constants). These can be found, as demonstrated in Section 7, by increasing ε and/or p. This way of improving the out-of-control performance sacrifices some in-control performance. Another way of improving the out-of-control performance is to increase m. For example, looking at Table 5 at ($p = 0.10,\ k = 0.50,\ ARL_0 = 200$) it can be seen that increasing m decreases the adjusted constants from *6.64* ($m = 30$) to *4.60* ($m = 200$). Note that, increasing m improves the out-of-control performance without sacrificing any in-control performance.

The issue of course is how to increase m given a small initial m such as $m = 30$. To solve this problem, we suggest first using $h^* = 6.64$ ($m = 30$), as in Fig. 4. Charting should continue until twenty subgroups, which plot within the adjusted control limits, are obtained. These 20 additional subgroups should be in-control and can be combined with the $m = 30$ Phase I subgroups. The resulting 50 subgroups can then be used to update the Phase I parameter estimates, which should now be used with the $h^* = 5.61$ ($m = 50$) limits. This updating process can be repeated until 200 in-control subgroups have been collected. At this point the out-of-control performance of the adjusted limits will be nearly equal to that of the unadjusted limits and the in-control performance will be as nominally specified.

8. Summary and conclusions

Currently, the in-control conditional average run length ($CARL_{IN}$) distribution of the two-sided Phase II CUSUM charts has been described using the mean and standard deviation. This $CARL_{IN}$ distribution and its parameters has been calculated by Markov Chains and simulations. We presented in this paper a generalization of the Siegmund formula to calculate the $CARL_{IN}$ distribution and its parameters. It has been seen that the generalization of the Siegmund formula is accurate, versatile and practical.

We argued in favor of the usefulness of the exceedance probability criterion based $CARL_{IN}$ prediction bounds to study the effects of parameter estimation. Based on these prediction bounds, it was seen that even more Phase I data are required than previously recommended using the $SDARL_{IN}$ criteria.

When analytical methods cannot be found, the bootstrap method has been used to adjust the control limits according to the exceedance probability criterion. In this paper we applied a more accurate method of Diko et al. (2019) to adjust the control limits according to the exceedance probability criterion. Comprehensive tables of the adjusted charting constants are given to facilitate implementation of the two sided CUSUM control chart for the mean.

The generalized Siegmund formula, presented in this paper, can also be used to find the unconditionally adjusted control limits. This has not been done in literature. Furthermore, our work can be extended to

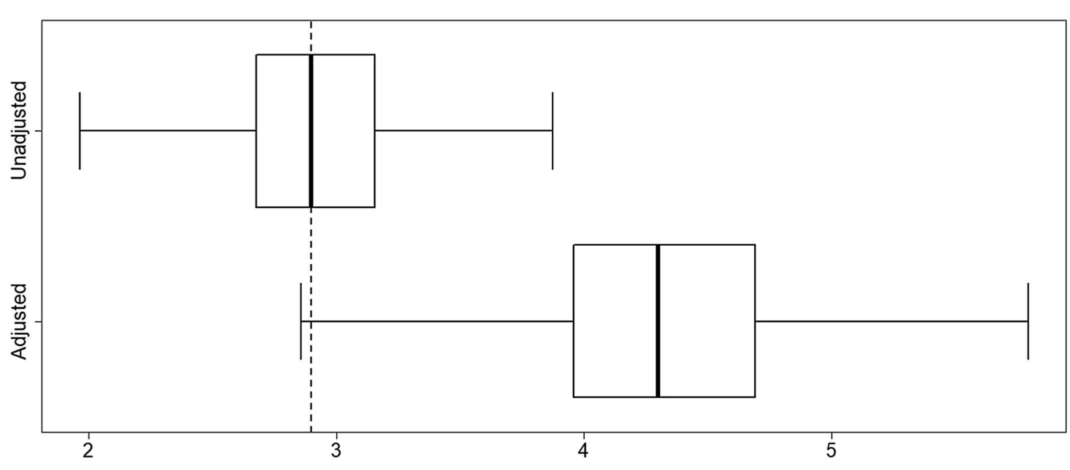

Fig. 6. Boxplots of the out-of-control *CARL* distribution of the two sided Phase II CUSUM chart for the mean of the hard bake process data.

other types of standard deviation estimators and process distributions.

Declaration of Competing Interest

The authors declared that there is no conflict of interest.

Acknowledgements

The authors would like to thank the editor, associate editor and anonymous reviewers for the comments and suggestions that have led to various improvements.

Appendix A. Simulation of the empirical distribution of *CARL* and the application of the Markov Chain and modified Siegmund formula approaches

Code to apply the Markov Chain Approach

```
CARL = function(br,k,h,delta,m,n){
if (n > 1) {ab = c(1,m * (n − 1))} else
{
M = function(m){
(0.8264 * m − 1.082)/(m − 1)^2}
r = function(m){
(−2 + 2 * sqrt(1 + 2 * M(m)))^−1}
t = function(m){
M(m) + 1/(16 * r(m)^3)}
bb = function(m){
(−2 + 2 * sqrt(1 + 2 * t(m)))^−1}
aa = function(m){
1 + (1/(4 * bb(m))) + (1/(32 * bb(m)^2)) − (5/(128 * bb(m)^3))}
ab = c(aa(m),bb(m))} # calculates the constants a and b as a function of m and n
Q = ab[1] * sqrt(rchisq(br,ab[2])/(ab[2]))
Z = rnorm(br)
t = 101 # the total number of transient states
w = h/t # the width of the transient states
sigma = 1 # assume sigma0
z = seq(1,t,1) # a vector of transient state identifiers
M = numeric(length(t)) # storage vector of the transient state midpoints
for(j in 1:t){
M[j] = z[j] * w − 0.5 * w} # midpoint of the jth transient state
Y = matrix(0,1,t + 1) # a size t vector of zeros
Y[1] = 1 # change the first element of Y from zero to one
P = diag(t + 1) # a t*t storage matrix of transient probabilities
I = diag(t + 1) # identity matrix
Q = sqrt(rchisq(br,m * (n − 1))/(m * (n − 1))) # estimation error for sigma
Z = rnorm(br) # estimation error for mu
for(j in 1:t){
for(l in 1:t){
A = Q * (M[j] + (w/2) − M[l] + k) − (((sqrt(n) * delta)/sigma)) + (Z/sqrt(m))
B = Q * (M[j] − (w/2) − M[l] + k) − (((sqrt(n) * delta)/sigma)) + (Z/sqrt(m))
P[l + 1,j + 1] = pnorm(A) − pnorm(B)}} # calculates the transient probabilities for j,l = 2,3,…,t
P[1,1] = pnorm(Q * (k) − (((sqrt(n) * delta)/sigma)) + (Z/sqrt(m))) # calculates the transient probabilities for j,l = 1
for(j in 1:t){
P[j + 1,1] = pnorm(Q * (−M[j] + k) − (((sqrt(n) * delta)/sigma)) + (Z/sqrt(m)))} # calculates the transient probabilities for j = 2,3,4,…,t and l = 1
for(l in 1:t){
P[1,l + 1] = pnorm(Q * (M[l] + (w/2) + k) − (((sqrt(n) * delta)/sigma)) + (Z/sqrt(m))) − pnorm(Q * (M[l] − (w/2) + k) − (((sqrt(n) * delta)/sigma)) + (Z/sqrt(m)))} # calculates the transient probabilities for j = 1 and l = 2,3,4,…,t
ARL1 = Y% * %solve(I − P)% * %matrix(1,t + 1,1) # CARL of the upper one-sided CUSUM
for(j in 1:t){
for(l in 1:t){
A = Q * (M[j] + (w/2) − M[l] + k) − (−((sqrt(n) * delta)/sigma)) + (−Z/sqrt(m))
B = Q * (M[j] − (w/2) − M[l] + k) − ( − ((sqrt(n) * delta)/sigma)) + (−Z/sqrt(m))
P[l + 1,j + 1] = pnorm(A) − pnorm(B)}} # calculates the transient probabilities for j,l = 2,3,…,t
P[1,1] = pnorm(Q * (k) − (−((sqrt(n) * delta)/sigma)) + (−Z/sqrt(m))) # calculates the transient probabilities for j,l = 1
for(j in 1:t){
P[j + 1,1] = pnorm(Q * (−M[j] + k) − (−((sqrt(n) * delta)/sigma)) + (−Z/sqrt(m)))} # calculates the transient probabilities for j = 2,3,4,…,t and l = 1
for(l in 1:t){
P[1,l + 1] = pnorm(Q * (M[l] + (w/2) + k) − (−((sqrt(n) * delta)/sigma)) + (−Z/sqrt(m))) − pnorm(Q * (M[l] − (w/2) + k) − (−((sqrt(n) * delta)/sigma)) + (−Z/sqrt(m)))} # calculates the transient probabilities for j = 1 and l = 2,3,4,…,t
ARL2 = Y% * %solve(I − P)% * %matrix(1,t + 1,1) # CARL of the lower one-sided CUSUM
((1/ARL1) + (1/ARL2))^−1} # CARL for the two-sided CUSUM (Van Dobben de Bruyn, 1968)
```

Note that, the codes for the transient state probabilities are based on the mathematical formulas that were provided in Saleh et al. (2016).

Code to apply the Modified Siegmund formula

```
CARL = function(br,k,h,delta,m,n){
if (n > 1) {ab = c(1,m * (n − 1))} else
{M = function(m){
(0.8264 * m − 1.082)/(m − 1)ˆ2}
r = function(m){
(−2 + 2 * sqrt(1 + 2 * M(m)))ˆ−1}
t = function(m){
M(m) + 1/(16 * r(m)ˆ3)}
bb = function(m){
(−2 + 2 * sqrt(1 + 2 * t(m)))ˆ−1}
aa = function(m){
1 + (1/(4 * bb(m))) + (1/(32 * bb(m)ˆ2)) − (5/(128 * bb(m)ˆ3))}
ab = c(aa(m),bb(m))} # calculates the constants a and b as a function of m and n
Q = ab[1] * sqrt(rchisq(br,ab[2])/(ab[2]))
Z = rnorm(br)

   ARL1 = (exp(2 * (−((sqrt(n) * delta)/sigma) + (Z/sqrt(m)) + k * Q) * (h * Q + 1.166)) − 2 * (−((sqrt(n) * delta)/sigma) + (Z/sqrt(m)) + k * Q) * (h * Q + 1.166) − 1)/(2 * (−-
   ((sqrt(n) * delta)/sigma) + (Z/sqrt(m)) + k * Q)ˆ2)

   ARL2=(exp(2 * (−(−(sqrt(n) * delta)/sigma) − (Z/sqrt(m)) + k * Q) * (h * Q + 1.166)) − 2 * (−(−(sqrt(n) * delta)/sigma) − (Z/sqrt(m)) + k * Q) * (h * Q + 1.166) − 1)/(2-
   * (−(−(sqrt(n) * delta)/sigma) − (Z/sqrt(m)) + k * Q)ˆ2)
((1/ARL1) + (1/ARL2))ˆ−1} # CARL for the two-sided CUSUM (Van Dobben de Bruyn, 1968)
```

Simulation of the distribution of the CARL

Given that *(k = 0.50,h = 4.172,*δ = 0,*m = 800,n = 5)*, run the following statement

CARLD = replicate(3000,CARL(1,k = 0.50,h = 4.172,delta = 0,m = 800,n = 5))

where h = xcusum.crit(k = 0.50,ARL_0 = 200,sided = “two”) and CARL(br = 1,k = 0.50,h = 4.172,delta = 0,m = 800,n = 5) is a random *CARL* value, which is calculated either by the Markov Chain approach or the modified Siegmund approach. The 3000 *CARL* values are stored in the vector *CARLD*.

The ordered (ascending) *CARLD* values and their associated relative frequency constitute an empirical *CARL* distribution, denoted by F_N.

Code to Calculate $AARL_{IN}$ by integrating the modified Siegmund formula

```
library(cubature)
AARL = function(k,h,delta,m,n){
if (n > 1) {ab = c(1,m * (n − 1))} else
{M = function(m){
(0.8264 * m − 1.082)/(m − 1)ˆ2}
r = function(m){
(−2 + 2 * sqrt(1 + 2 * M(m)))ˆ−1}
t = function(m){
M(m) + 1/(16 * r(m)ˆ3)}
bb = function(m){
(−2 + 2 * sqrt(1 + 2 * t(m)))ˆ−1}
aa = function(m){
1 + (1/(4 * bb(m)))+(1/(32 * bb(m)ˆ2)) − (5/(128 * bb(m)ˆ3))}
ab = c(aa(m),bb(m))} # calculates the constants a and b as a function of m and n

   CARL1 = function(x){(exp(2 * (−((sqrt(n) * delta)/sigma) + (x[1]/sqrt(m)) + k * (ab[1] * sqrt(x[2]/(ab[2])))) * (h * (ab[1] * sqrt(x[2]/(ab[2]))) + 1.166)) − 2 * (−((sqrt(n) * -
   delta)/sigma) + (x[1]/sqrt(m)) + k * (ab[1] * sqrt(x[2]/(ab[2])))) * (h * (ab[1] * sqrt(x[2]/(ab[2]))) + 1.166) − 1)/(2 * (−((sqrt(n) * delta)/sigma) + (x[1]/sqrt(m)) + k * (ab
   [1] * sqrt(x[2]/(ab[2]))))ˆ2)}
CARL2 = function(x){
   (exp(2 * (−(−(sqrt(n) * delta)/sigma) − (x[1]/sqrt(m)) + k * (ab[1] * sqrt(x[2]/(ab[2])))) * (h * (ab[1] * sqrt(x[2]/(ab[2]))) + 1.166)) − 2 * (−(−(sqrt(n) * delta)/sigma) − (-
   x[1]/sqrt(m)) + k * (ab[1] * sqrt(x[2]/(ab[2])))) * (h * (ab[1] * sqrt(x[2]/(ab[2]))) + 1.166) − 1)/(2 * (−(−(sqrt(n) * delta)/sigma) − (x[1]/sqrt(m)) + k * (ab[1] * sqrt(x[2]/
   (ab[2]))))ˆ2)}
CARL = function(x){ (((1/CARL1(x)) + (1/CARL2(x)))ˆ−1)}
ARL = function(x){CARL(x) * dnorm(x[1],0,1) * dchisq(x[2],ab[2])}
b = qchisq(0.99999,ab[2])
adaptIntegrate(ARL,c(−100,0),c(100,b),tol = 1e−10)[[1]]}
```

For example, running AARL(k = 0.25,h = 6.854,delta = 0,m = 1000,n = 5) gives the answer 194

Appendix B. Algorithm for finding the $CARL_{IN,p}$ and *%RE*

Step 1: Fix *m, n, k, h,* p_0 and ARL_0.
Step 2: Generate the empirical distribution of $CARL_{IN}$ (See Appendix A).
Step 3: Calculate the 100*p*th percentile $CARL_{IN,p}$ of F_N.
Step 4: Calculate *PD*, see Eq. (14).

Step 5: Interpret PD. A negative $\%RE$ value means that $CARL_{IN,p} < ARL_0$ by PD percentage points. A positive PD value means that $CARL_{IN,p} > ARL_0$ by PD percentage points.

Appendix C. A step by step algorithm for finding the minimum *m* required to design a two-sided Phase II CUSUM control chart according to exceedance probability criterion

Step 1: Fix p, ARL_0, ε, h, n, k and a starting value of m.
Step 2: Generate F_N the empirical distribution of $CARL_{IN}$(See Appendix A).
Step 3: Calculate $CARL_{IN,p}$.
Step 4: If $CARL_{IN,p} > ARL_0(1 - \varepsilon)$ stop and use the current value of m otherwise increment m and return to step 2.

Appendix D. A step by step algorithm for finding a suitable *h* via our method

Step 1: Specify p, ARL_0, ε, m, n, k and a search interval for h. The lower bound of the search interval should be the value of h when the parameters are known (Case K), (See also Table 1.).
Step 2: Generate the empirical distribution of $CARL_{IN}$ (See Appendix A).
Step 3: Calculate $CARL_{IN,p}$.
Step 4: If $CARL_{IN,p} > ARL_0(1 - \varepsilon)$ stop and use the current h as an adjusted limit otherwise increment h by 0.1 and return to step 2.

Appendix E. A step by step algorithm for finding a suitable *h* via parametric bootstrapping

Step 1: Specify p, ARL_0, ε, m, n, k and a search interval for h.
Step 2: Generate m samples of size n from $N(\mu_0, \sigma_0)$.
Step 3: Calculate $\hat{\mu}_0$ and $\hat{\sigma}_{0g}$.
Step 4: Generate m samples of size n from $N(\hat{\mu}_0, \hat{\sigma}_{0g})$.
Step 5: Calculate $\mu_0^*$and σ_{0g}^* the same way as $\hat{\mu}_0$ and $\hat{\sigma}_{0g}$.
Step 6: For each value of h in the specified interval of h, use Markov Chains to calculate $CARL_{IN}$. Assume that the chart limits are constructed from the Phase I estimates (μ_0^*, σ_0^*) and that the in-control distribution of $\bar{X}_i$ is $N(\mu_0^*, \sigma_{0g}^*/\sqrt{n})$.
Step 7: Select the value of h (say h^*) that satisfies the condition $CARL_{IN} = ARL_0$.
Step 8: Repeat the Steps (4) to (7) B times (e.g. $B = 1000$ times).
Step 9: Order the h^* values in ascending order.
Step 10: The adjusted limit is the $(1 - p)^{th}$ percentile of the ordered h^*values.

Appendix F. The R code for the search algorithm of *h*

```
m = 25   # number of subgroups
n = 5   # sample size
ICARL = 200   # in-control average runlength
epc = 0.10   # exceedance probability
e = 0   # percentage relative error
k = 0.50   # depends on anticipated shift
h = seq(from = 6.64, to = 1000, by = 0.01) # search interval with lower bound 6.64
for(i in 1:length(h)){
dd = replicate(500,000,CARL(1,k,h[i],0,m,n)) # the function CARL is given in Appendix A
QQ = quantile(dd,p = c(epc))
if (QQ > ICARL-e*ICARL) break
}
h[i]
```

Appendix G. Acronyms

Abbreviation	Description
ARL_{IN}	in-control average run length
ARL_0	nominal in-control average run length
$CARL$	conditional average run length
$CARL_{IN}$	conditional in-control average run length
$AARL_{IN}$	unconditional in-control average run length
$SDARL_{IN}$	standard deviation of the conditional in-control average run length
$CARL_{IN,p}$	100pth percentile of the conditional in-control average run length
RE	relative error
PD	percentage difference
m	Number of Phase I subgroups of size n

References

Abbasi, S. A., Riaz, M., & Miller, A. (2012). Enhancing the performance of CUSUM scale chart. *Computers & Industrial Engineering, 63*, 400–409.

Albers, W., & Kallenberg, W. C. M. (2004). Estimation in Shewhart control charts: Effects and corrections. *Metrika, 59*(3), 207–234.

Brook, D., & Evans, D. A. (1972). An approach to the probability distribution of CUSUM run length. *Biometrika, 59*(3), 539–549.

Chakraborti, S. (2000). Run Length, average run length and false alarm rate of Shewhart X-bar chart: Exact derivations by conditioning. *Journal of Applied Statistics, 33*(4), 439–459.

Chakraborti, S. (2006). Parameter estimation and design considerations in prospective applications of the X Chart. *Communications in Statistics: Simulation and Computation, 29*(1), 61–81.

Diko, M. D., Chakraborti, S., & Does, R. J. M. M. (2019). Guaranteed in-control performance of the EWMA chart for monitoring the mean. *Quality and Reliability Engineering International, 35*(4), 1144–1160.

Diko, M. D., Chakraborti, S., & Graham, M. A. (2016). Monitoring the process mean when standards are unknown: A classic problem revisited. *Quality and Reliability Engineering International, 32*(2), 609–622.

Diko, M. D., Goedhart, R., Chakraborti, S., Does, R. J. M. M., & Epprecht, E. K. (2017). Phase II control charts for monitoring dispersion when parameters are estimated. *Quality Engineering, 29*(4), 605–622.

Epprecht, E. K., Loureiro, L. D., & Chakraborti, S. (2015). Effect of the amount of phase I data on the phase II performance of S^2 and S control charts. *Journal of Quality Technology, 47*(2), 139–155.

Gandy, A., & Kvaloy, J. T. (2013). Guaranteed conditional performance of control charts via bootstrap methods. *Scandinavian Journal of Statistics, 40*, 647–668.

Goedhart, R., da Silva, M. M., Schoonhoven, M., Epprecht, E. K., Chakraborti, S., Does, R. J. M. M., & Veiga, A. (2017). Shewhart control charts for dispersion adjusted for parameter estimation. *IISE Transactions, 49*(8), 838–848.

Goedhart, R., Schoonhoven, M., & Does, R. J. M. M. (2016). Correction factors for Shewhart X and X-bar control charts to achieve desired unconditional ARL. *International Journal of Production Research, 54*(24), 7464–7479.

Goedhart, R., Schoonhoven, M., & Does, R. J. M. M. (2017). Guaranteed in-control performance for the Shewhart X and X-bar control charts. *Journal of Quality Technology, 49*(2), 155–171.

Goedhart, R., Schoonhoven, M., & Does, R. J. M. M. (2018). Letter to the editor: On guaranteed in-control performance for the Shewhart X and X-bar control charts. *Journal of Quality Technology, 50*(1), 130–132.

Hu, X., & Castagliola, P. (2017). Guaranteed conditional design of the median chart with estimated parameters. *Quality and Reliability Engineering International, 33*(8), 1873–1884.

Jardim, F. S., Chakraborti, S., & Epprecht, E. K. (2018). Two perspectives for designing a Phase II control chart with estimated parameters: The Case of the Shewhart X chart. *Journal of Quality Technology*. https://doi.org/10.1080/00224065.2019.1571345 (in early view).

Jeske, D. R. (2016). Determining the reference sample size needed to control the accuracy of the conditional in-control ARL of a normal theory CUSUM. *Quality and Reliability Engineering International, 32*(7), 2499–2504.

Jones, L. A., Champ, C. W., & Rigdon, S. E. (2004). The run length distribution of the CUSUM with estimated parameters. *Journal of Quality Technology, 36*(1), 95–108.

Jones, M. A., & Steiner, S. H. (2012). Assessing the effect of estimation error on the risk-adjusted CUSUM chart performance. *International Journal for Quality in Health Care, 24*(2), 176–181.

Loureiro, L. D., Epprecht, E. K., Chakraborti, S., & Jardim, F. S. (2018). In-control performance of the joint Phase II Xbar-S control charts when parameters are estimated. *Quality Engineering, 30*(2), 253–267.

Lucas, J. M., & Crosier, R. B. (1982). Fast initial response for CUSUM quality control schemes. *Technometrics, 24*, 199–205.

Mahmoud, M. A., Henderson, G. R., Epprecht, E. K., & Woodall, W. H. (2010). Estimating the standard deviation in quality-control applications. *Journal of Quality Technology, 42*(4), 348–357.

Montgomery, D. C. (2013). *Introduction to statistical quality control: A modern introduction* (7th ed.). Hoboken: John Wiley & Sons.

Patnaik, P. B. (1950). The use of mean range as an estimator of variance in statistical tests. *Biometrika, 37*, 78–87.

Saleh, N. A., Zwetsloot, I. M., Mahmoud, M. A., & Woodall, W. H. (2016). CUSUM charts with controlled conditional performance under estimated parameters. *Quality Engineering, 28*(4), 402–415.

Sanusi, R. A., Abujiya, M. R., Riaz, M., & Abbas, N. (2017). Combined Shewhart CUSUM charts using auxiliary variable. *Computers & Industrial Engineering, 105*, 329–337.

Siegmund, D. (1985). *Sequential analysis: Tests and confidence intervals.* Berlin: Springer.

Van Dobben de Bruyn, C. S. (1968). *Cumulative sum tests: Theory and practice.* London: Griffin.

Woodall, W. H. (1984). On the Markov Chain approach to the two-sided CUSUM procedure. *Technometrics, 26*, 41–46.