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Mixed Cumulative Sum–Exponentially Weighted Moving Average Control Charts: An Efficient Way of Monitoring Process Location

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Shewhart, exponentially weighted moving average (EWMA), and cumulative sum (CUSUM) charts are famous statistical tools, to handle special causes and to bring the process back in statistical control. Shewhart charts are useful to detect large shifts, whereas EWMA and CUSUM are more sensitive for small to moderate shifts. In this study, we propose a new control chart, named mixed CUSUM-EWMA chart, which is used to monitor the location of a process. The performance of the proposed mixed CUSUM-EWMA control chart is measured through the average run length, extra quadratic loss, relative average run length, and a performance comparison index study. Comparisons are made with some existing charts from the literature. An example with real data is also given for practical considerations. Copyright © 2014 John Wiley & Sons, Ltd.

Keywords: average run length; Shewhart; cumulative sum; exponentially weighted moving average; statistical process control

1. Introduction

There are two major types of variations in processes that affect the product characteristics: one is special cause variation and the other is common cause variation. A process is considered in control in the presence of only common cause variations, but the presence of special cause variations brings it out of control. Control charts are famous tools to differentiate between these two states of a process (Shewhart[1]). Shewhart control charts are mostly used to detect large shifts in location and/or dispersion parameters. On the other hand, the exponentially weighted moving average (EWMA) control chart and the cumulative sum (CUSUM) control chart are popular for small to moderate shifts (cf. Roberts[2] and Page,[3] respectively).

There is a variety of literature on these types of charts for efficient monitoring of process parameters and improving the quality of the process outputs. In order to enhance the detection abilities of different kinds of charts, researchers have suggested certain modifications in the literature. Lucas[4] proposed a combined Shewhart-CUSUM quality control scheme for efficient detection of small and large shifts. Lucas and Saccucci[5] recommended a combined Shewhart-EWMA control chart for improved performance. Lucas and Crosier[6] proposed fast initial response (FIR) CUSUM charts that provide a head start to the CUSUM statistics, and similarly, Steiner[7] proposed FIR EWMA. Yashchin[8] proposed the weighted CUSUM in which he assigned weights to the past information in CUSUM statistics. Riaz *et al.*[9] used the idea of runs rules to enhance the performance of the CUSUM control charts for small to large shifts. Riaz *et al.*[10] implemented different runs rules schemes, and Mehmood *et al.*[11] used a variety of ranked set strategies to enhance performance of Shewhart charts. Abbas *et al.*[12] applied the runs rules idea for the EWMA charts. Recently, Abbas *et al.*[13] introduced the design structure of a mixed EWMA-CUSUM (MEC) control chart for improved monitoring of the process parameters. In the said MEC chart, the EWMA statistic is used as the input for the CUSUM structure. In this study, we propose a reverse version of this mixing, that is, a mixed CUSUM-EWMA (MCE) control chart. In this new setup, the CUSUM statistic will serve the input for the EWMA structure.

The organization of the rest of this study is as follows: Section 2 describes the classical CUSUM and classical EWMA control charts, the MEC of Abbas *et al.*,[13] and the proposed scheme denoted by MCE; section 3 contains the explanations of the different measures that will be used to evaluate the performance, the comparisons using these performance measures, and some graphical presentations; in section 4, an example with a real data set is given for the practical aspects of the proposed scheme; finally, we conclude the article with a Section 5.

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2. Cumulative sum (CUSUM), exponentially weighted moving average (EWMA), mixed EWMA-CUSUM, and the proposed mixed CUSUM-EWMA control charts

Cumulative sum and EWMA charts are designed to use the previous information along with the current to detect small to moderate shifts. Consequently, they are known as memory control charts. Abbas *et al.*[13] proposed a new memory control chart, by mixing the features of the EWMA and CUSUM control charts (the chart is denoted by MEC). Following them, we propose the reverse order of their mixture, that is, the CUSUM statistic is used as input for the EWMA chart. The resulting chart is denoted by the MCE control chart. The description of all these charts (i.e., CUSUM, EWMA, MEC, and MCE) is given in the following subsections.

2.1. *Classical cumulative sum scheme*

The CUSUM chart was introduced by Page,[3] and it uses the cumulative deviation from the target value. It is a favorable tool to detect small to moderate shifts. The CUSUM chart is based on the following two statistics:

$$\left.\begin{aligned} C_t^+ &= \max\left[0, \left(\overline{X}_t - \mu_0\right) - K + C_{t-1}^+\right] \\ C_t^- &= \max\left[0, -\left(\overline{X}_t - \mu_0\right) - K + C_{t-1}^-\right] \end{aligned}\right\} \tag{1}$$

where t represents the time or sample number and $\overline{X}_t$ is the mean of X for sample t, $X_t \sim N(\mu_0, \sigma_0)$, where μ_0 and σ_0 are the target mean and standard deviation, respectively. $K = k\sigma_o$ is the reference value and is mostly used half of the shift, that is, $k = \frac{\delta}{2}$, where δ is the amount of shift given as $\delta = \frac{|\mu_1 - \mu_0|}{\sigma_0/\sqrt{n}}$, μ_1 is the out of control mean, and n is the sample size. C_t^+ and C_t^- are the upper and lower CUSUM statistics, respectively, and are plotted against to the control limit $H = h\sigma_o$. Initially, we set $C_0^+ = C_0^- = 0$.

2.2. *Classical exponentially weighted moving average scheme with time-varying limits*

The EWMA control chart was introduced by Roberts,[2] and it is also used to detect small to moderate shifts (like the CUSUM chart). The EWMA statistic is defined as follows:

$$Z_t = (1-\lambda)Z_{t-1} + \lambda\overline{X}_t \tag{2}$$

where λ is the sensitivity parameter with $0 < \lambda \le 1$. Z_0 is the starting value and is set to be equal to the target mean μ_0. The EWMA structure has an upper control limit (UCL), lower control limit (LCL), and center line (CL) defined as follows:

$$\left.\begin{aligned} LCL_t &= \mu_0 - L\sigma_{\overline{X}}\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)} \\ CL &= \mu_0 \\ UCL_t &= \mu_0 + L\sigma_{\overline{X}}\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)} \end{aligned}\right\} \tag{3}$$

where L is used as width coefficient between UCL and LCL for the predefined false alarm rate. Steiner[7] explained the details of the EWMA performance with time-varying limits used to monitor the location of the normally distributed process.

2.3. *Mixed exponentially weighted moving average–cumulative sum scheme*

Abbas *et al.*[13] proposed a new chart by mixing the design structures of the classical EWMA and CUSUM charts. The plotting statistics of the EWMA in Equation (2) is used as input for the CUSUM chart, and hence, the plotting statistic for the MEC chart is given as follows:

$$\left.\begin{aligned} MEC_t^+ &= \max\left[0, (Z_t - \mu_0) - K_{z_t} + MEC_{t-1}^+\right] \\ MEC_t^- &= \max\left[0, -(Z_t - \mu_0) - K_{z_t} + MEC_{t-1}^-\right] \end{aligned}\right\} \tag{4}$$

where K_{z_t} is the time-varying reference value and is defined as follows:

$$K_{z_t} = k_Z \times \sqrt{\mathrm{Var}(Z)} = k_Z\sigma_{\overline{X}}\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)} \tag{5}$$

The MEC_t^+ and MEC_t^- statistics are the upper and the lower statistics, respectively, for the *MEC* control chart. Now, these statistics are plotted against the control limit H_{z_t}, which is defined as follows:

$$H_{z_t} = h_Z \times \sqrt{\mathrm{Var}(Z)} = h_Z\sigma_{\overline{X}}\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)} \tag{6}$$

where h_z is the coefficient used to fix the predefined false alarm rate. Any value of MEC^+ crossing the control limit H_{z_t} indicates an increase in the process mean, and if MEC^- goes beyond H_{z_t} for any value of t, then it will point out a negative shift in the process location.

2.4. The proposed mixed cumulative sum–exponentially weighted moving average scheme

The proposed scheme is also a mixture of the CUSUM and EWMA features but in the reverse order compared with the MEC control chart. The proposed MCE chart is based on two statistics, which are given as follows:

$$\left.\begin{aligned} MCE_t^+ &= (1-\lambda_C)MCE_{t-1}^+ + \lambda_C C_t^+ \\ MCE_t^- &= (1-\lambda_C)MCE_{t-1}^- + \lambda_C C_t^- \end{aligned}\right\} \tag{7}$$

where C_t^+ and C_t^- are the classical CUSUM statistics given in Equation (1) and λ_C is the sensitivity parameter of the proposed chart with $0 < \lambda_c \leq 1$. The initial values for the statistics given in Equation (7) are taken equal to the target mean of C_t^+ and C_t^-, respectively, that is, $MCE_0^+ = MCE_0^- = \mu_C$. For the in control situation, the mean and the variance of the statistics in Equation (7) are time varying up to a specific value of t, and for $t \to \infty$, they become constant. The notations for the mean and variance are given as follows:

$$\left.\begin{aligned} \text{Mean}\left(C_t^+\right) &= \text{Mean}\left(C_t^-\right) = \mu_{c_t} \\ \text{Var}\left(C_t^+\right) &= \text{Var}\left(C_t^-\right) = \sigma_{c_t}^2 \end{aligned}\right\} \tag{8}$$

Using the mean and variance from Equation (8), we define the control limit of the proposed chart as follows:

$$UCL_t = \mu_{c_t} + L_C\sigma_{c_t}\sqrt{\frac{\lambda_C}{2-\lambda_C}\left(1-(1-\lambda_C)^{2t}\right)} \tag{9}$$

where L_C is the width coefficient, like L in Equation (3), and determines the predefined false alarm rate. Any shift in the positive direction will be taken care by MCE_t^+, that is, if any value of MCE_t^+ crosses the control limit in Equation (9), the process mean will be declared as shifted upwards. Similarly, any shift in the negative direction will be addressed by MCE_t^-, that is, if any value of MCE_t^+ goes beyond the control limit, the process mean will be deemed as shifted downwards.

3. Performance measures evaluation and comparison

There are different measures, available in the literature, used to judge the performance of a control chart. Some of them are evaluated for a specific value of δ while others are calculated for a range of δ. A few of them is elaborated in the following lines. The average run length (*ARL*) is a famous tool and is widely used by researchers for measuring the performance of memory type control charts. The performance is assessed by two types of *ARL*s, that is, ARL_0 and ARL_1. ARL_0 is the expected number of samples before an out of control point is detected when the process is actually in control. ARL_1 is the expected number of samples before an out of control signal is received when the process is actually shifted to an out of control state. For a fixed value of ARL_0, a chart is considered to be more effective than other charts if it has a smaller ARL_1 (Wu *et al.*[14]) There are different approaches for evaluating the *ARL*. In the literature, we find methods such as Markov chains, integral equations, and Monte Carlo simulations. We have used Monte Carlo simulation to calculate the *ARL* measures for the proposed control chart. The algorithm is developed in MATLAB 7.1.2 and is run 10^4 times to obtain run lengths. Finally, these run lengths are averaged to obtain the *ARL* value. The *ARL*s for the proposed *MCE* chart are given in Table I.

The extra quadratic loss (*EQL*) is an alternative measure of the *ARL*. The *ARL* value evaluates the performance of a charting structure at a specific shift point, while *EQL* describes the overall effectiveness of a control chart for a range of values of the shift δ. It is defined as follows:

$$EQL = \frac{1}{\delta_{\max}-\delta_{\min}}\int_{\delta_{\min}}^{\delta_{\max}} \delta^2 ARL(\delta)d\delta \tag{10}$$

Hence, *EQL* is defined as a weighted average of *ARL*s over the whole process shift domain (i.e., $\delta_{\min} < \delta < \delta_{\max}$) using the square of the shift (δ^2) as a weight.

The relative ARL (*RARL*) describes the overall effectiveness of a particular charting structure relative to a benchmark chart. It examines how close a particular chart performs to the benchmark chart for each shift in terms of *ARL* (cf. Wu *et al.*[14] and Ryu *et al.*[15]).

$$RARL = \frac{1}{\delta_{\max}-\delta_{\min}}\int_{\delta_{\min}}^{\delta_{\max}} \frac{ARL(\delta)}{ARL_{\text{benchmark}}(\delta)}d\delta \tag{11}$$

The chart having the lowest values of *ARL* is generally considered to be the benchmark chart. The *RARL* may be observed as $RARL = 1$ for the benchmark chart and $RARL > 1$ for the other charts. The aforementioned *RARL* value 1 shows the inferiority in performance of a particular chart relative to the benchmark chart. For evaluating the *RARL*, Zhao *et al.*[16] and Han *et al.*[17] considered the uniform distribution of δ in their studies.

The performance comparison index (*PCI*), according to Ou *et al.*,[18] is the ratio between the *EQL* of a chart and the *EQL* of the best chart under the same conditions. This index facilitates the performance comparison and a ranking based on the *EQL*. The chart with the lowest *EQL* has a *PCI* value equal to one, and the *PCI* values of all other charts are larger than one.

Table I. Average run length (ARL) values for the proposed mixed cumulative sum–exponentially weighted moving average control chart

δ	$ARL_0 \cong 168$				$ARL_0 \cong 400$				$ARL_0 \cong 500$			
	$\lambda_C = 0.1$ $L_C = 5.96$	$\lambda_C = 0.25$ $L_C = 6.4$	$\lambda_C = 0.5$ $L_C = 5.4$	$\lambda_C = 0.75$ $L_C = 4.6$	$\lambda_C = 0.1$ $L_C = 8.9$	$\lambda_C = 0.25$ $L_C = 8.4$	$\lambda_C = 0.5$ $L_C = 6.932$	$\lambda_C = 0.75$ $L_C = 5.754$	$\lambda_C = 0.1$ $L_C = 9.66$	$\lambda_C = 0.25$ $L_C = 9.1$	$\lambda_C = 0.5$ $L_C = 7.43$	$\lambda_C = 0.75$ $L_C = 6.08$
0	170	169.39	172.06	171.28	407.03	396.46	395.65	395.3	501.99	500.45	501.71	502.89
0.25	67.79	70.42	72.9	73.45	112.54	117.63	121.53	122.56	127.63	135.89	143.86	142.74
0.5	25.61	24.94	25.61	25.43	33.66	33.98	34.53	35.17	36.03	37.65	38.25	37.66
0.75	13.52	12.77	12.64	12.43	16.72	15.82	15.76	15.51	17.61	17	16.64	16.42
1	9.34	8.39	7.78	7.54	11.2	10.11	9.44	9.27	11.81	10.73	10.04	9.63
1.25	7.07	6.29	5.63	5.33	8.55	7.45	6.73	6.43	8.96	7.87	7.16	6.71
1.5	5.87	5.05	4.42	4.09	7.01	6.01	5.28	4.87	7.3	6.3	5.51	5.07
1.75	5.02	4.31	3.66	3.29	6.01	5.04	4.33	3.95	6.27	5.3	4.52	4.09
2	4.41	3.73	3.14	2.79	5.28	4.43	3.7	3.28	5.49	4.63	3.88	3.44
2.25	3.94	3.32	2.76	2.45	4.74	3.96	3.24	2.84	4.95	4.12	3.39	2.95
2.5	3.59	3.01	2.49	2.15	4.3	3.55	2.89	2.5	4.49	3.73	3.02	2.63
3	3.07	2.54	2.12	1.78	3.67	3.02	2.43	2.09	3.83	3.16	2.52	2.18

Table II. Average run length (ARL) values for the classical exponentially weighted moving average (EWMA), cumulative sum (CUSUM), Shewhart, and mixed EWMA-CUSUM (MEC) control charts

	Classical EWMA				Classical CUSUM			MEC		Shewhart chart		
	$\lambda = 0.1$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$k = 0.5$	$k = 0.5$	$k = 0.5$	$\lambda = 0.1$, $k_Z = 0.5$	$\lambda = 0.25$, $k_Z = 0.5$			
δ	$L = 2.824$	$L = 3$	$L = 3.07$	$L = 3.09$	$h = 4$	$h = 4.85$	$h = 5.065$	$h_Z = 33.54$	$h_Z = 11.2$	L = 2.807	L = 3.023	L = 3.09
0	500.18	500.26	500.07	501.35	167.2	402.17	500.68	400	500.95	199.98	399.55	499.61
0.25	103.75	172.52	252.21	326.66	74.31	129.08	143.87	73.31	80.26	156.33	302.19	373.89
0.5	29.66	46.91	88.62	138.48	26.65	35.78	38.71	33.06	35.74	90.92	165.8	201.44
0.75	14.16	19.46	35.49	62.64	13.4	16.37	17.29	22.39	24.0.6	49.92	86.25	103.06
1	8.78	10.55	17.24	30.53	8.37	10.06	10.53	17.63	18.45	28.21	46.37	54.55
1.25	6.24	6.95	9.83	16.49	6.05	7.15	7.49	9.23	15.45	16.73	26.23	30.4
1.5	4.76	5.11	6.37	10.05	4.77	5.62	5.77	12.88	3.79	10.46	15.65	17.88
1.75	3.83	3.98	4.54	6.47	3.35	3.9	4.04	7.89	12.34	6.88	9.85	11.1
2	3.18	3.26	3.52	4.57	2.45	2.33	3.67	10.45	11.19	4.77	6.53	7.25

Table III. Performance measures (overall) of the proposed mixed cumulative sum–exponentially weighted moving average (MCE) and other comparative schemes

$k_c = 0.5$	MCE	Classical EWMA	Classical CUSUM	MCE	Runs rules EWMA 1	MCE	Runs rules EWMA 2	MCE	Classical CUSUM	MCE	Classical CUSUM	MCE	Classical CUSUM
	$ARL_0 \cong 500$			$ARL_0 \cong 500$		$ARL_0 \cong 500$		$ARL_0 \cong 500$		$ARL_0 \cong 400$		$ARL_0 \cong 168$	
$\lambda_c = 0.1$													
EQL	12.4	8.65	10.79	12.43	8.90	12.4	7.70	12.41	10.79	11.77	10.25	9.51	8.27
RARL	1.37	1.00	1.23	1.34	1.00	1.59	1.00	1.21	1.08	1.22	1.09	1.22	1.10
PCI	1.43	1.00	1.24	1.39	1.00	1.61	1.00	1.26	1.09	1.27	1.10	1.29	1.12
$\lambda_c = 0.25$													
EQL	11.29	10.65	10.79	11.31	10.86	11.3	8.79	11.29	10.79	10.56	10.25	8.53	8.27
RARL	1.04	1.00	1.00	1.02	1.00	1.26	1.00	1.12	1.08	1.11	1.09	1.12	1.10
PCI	1.06	1.00	1.01	1.04	1.00	1.28	1.00	1.14	1.09	1.14	1.10	1.16	1.12
$\lambda_c = 0.5$													
EQL	10.41	15.76	10.79	10.42	14.57	10.4	10.54	10.41	10.79	9.70	10.25	7.77	8.27
RARL	1.00	1.55	1.03	1.00	1.43	1.00	1.02	1.05	1.08	1.03	1.09	1.04	1.10
PCI	1.00	1.51	1.03	1.00	1.39	1.00	1.01	1.06	1.09	1.04	1.10	1.05	1.12
$\lambda_c = 0.75$													
EQL	9.82	24.85	10.79	9.83	19.34	9.83	12.58	9.82	10.79	9.26	10.25	7.35	8.27
RARL	1.00	2.52	1.08	1.00	1.97	1.00	1.27	1.00	1.08	1.00	1.09	1.00	1.10
PCI	1.00	2.52	1.09	1.00	1.96	1.00	1.28	1.00	1.09	1.00	1.10	1.00	1.12

EQL, extra quadratic loss; RARL, relative average run length; PCI, performance comparison index.

Table IV. Average run length values of the runs rules exponentially weighted moving average (EWMA) and fast initial response (FIR) cumulative sum (CUSUM) schemes

	Runs rules-based EWMA scheme 1				Runs rules-based EWMA scheme 2				FIR CUSUM	
	$\lambda = 0.1$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 0.1$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$h = 5$	$h = 5$
δ	$L = 2.556$	$L = 2.554$	$L = 2.35$	$L = 2.11$	$L = 2.3$	$L = 2.345$	$L = 2.202$	$L = 1.982$	$C_0 = 1$	$C_0 = 2$
0	501.75	500.53	501.26	502.07	502	499.61	505.35	501.96	163	149
0.25	103.31	169.13	235.11	280.61	66.68	97.01	133.71	155.70	71.1	62.7
0.5	29.57	47.01	78.07	108.87	21.42	31.20	46.35	57.77	24.4	20.1
0.75	14.32	19.27	30.87	45.34	11.74	14.42	20.62	26.03	11.6	8.97
1	8.95	10.59	15.19	22.10	7.55	8.67	11.09	13.83	7.04	5.29
1.5	4.911	5.25	6.10	7.78	4.46	4.70	5.13	5.78	3.85	2.86
2	3.44	3.55	3.68	4.08	3.45	3.54	3.62	3.77	2.7	2.01

$$PCI = \frac{EQL}{EQL_{\text{benchmark}}} \tag{12}$$

Some modifications of these measures may be seen in Zhang and Wu,[19] Wu *et al.*,[14] Ryu *et al.*,[15] Ou *et al.*,[20] Ou *et al.*,[18] Ahmad *et al.*,[21] Ahmad *et al.*,[22] and Ahmad *et al.*[23].

The next subsections will provide a comprehensive comparison of our proposed chart with existing control charts from the literature. These are the classical EWMA by Roberts,[2] the classical CUSUM by Page,[3] the runs rules-based EWMA proposed by Abbas *et al.*[12] and runs rules-based CUSUM by Riaz *et al.*,[9] the FIR CUSUM by Lucas and Crosier,[6] weighted CUSUM scheme is proposed by Yashchin,[8] and, finally, the *MEC* chart by Abbas *et al.*[13]. For our study, we have used the following values of the chart parameters: $k = 0.5$ (cf. Equation (1)) and $\lambda_C = 0.1$, 0.25, 0.5, and 0.75 (cf. Equation (2)). For the ARL performance of the CUSUM chart, we have reproduced the results of Hawkins[24] and Montgomery[25] and found similar results.

3.1. Proposed mixed cumulative sum–exponentially weighted moving average (EWMA) versus time-varying EWMA

The *ARL* values of classical EWMA with time-varying limits (cf. Roberts[2]) are provided in Table II. The proposed scheme is compared with the classical EWMA scheme at different values of λ_c, while k is fixed. We observe that the proposed scheme has better ARL_1 performance for different λ_c values and for different shifts δ. For example, when $\lambda_c = 0.25$, the *MCE* has a better performance for $\delta \leq 1$, whereas for $\lambda_c = 0.5$, the proposed scheme seems superior for $\delta \leq 1.75$. Similarly, for $\lambda_c > 0.5$, the proposed MCE chart has smaller ARL_1 values for the entire range of δ (cf. Table II vs. Table I).

Similarly, the overall performance of the charts is judged by the EQL, RARL, and PCI measures. The proposed scheme at $\lambda \geq 0.5$ has efficient performance as compared with the classical EWMA because it has the lowest values of EQL, RARL, and PCI (cf. Table III).

3.2. Proposed mixed cumulative sum–exponentially weighted moving average (EWMA) versus run rules-based EWMA

The runs rules-based EWMA schemes are proposed by Abbas *et al.*,[12]and their *ARL*s values are given in Table IV. The *ARL* performance of the proposed scheme is compared with the runs rules-based EWMA schemes at different values of λ_c with its corresponding λ. The runs rules EWMA scheme 1 at $\lambda = 0.1$ has better performance than our proposed scheme, but at $\lambda_c = 0.25$, the proposed MCE scheme is performing better for $\delta \leq 0.75$. As the values of λ_c increases (i.e., $\lambda_c \geq 0.5$), the proposed MCE scheme is really performing better over runs rules EWMA scheme 1 for different shifts. The runs rules-based EWMA scheme 2 is bit superior when $\lambda \leq 0.25$, but for others values λ(i.e., $\lambda \geq 0.5$), the proposed scheme is superior for $\delta \leq 1.25$ (cf. Table IV vs. Table I).

For the EQL, RARL, and PCI measures, the proposed scheme is compared with the runs rules EWMA schemes 1 and 2 separately (cf. Table III). The proposed MCE scheme is more efficient than the runs rules EWMA schemes for $\lambda_c \geq 0.5$ because it has smaller EQL, RARL, and PCI values.

3.3. Proposed mixed cumulative sum (CUSUM)–exponentially weighted moving average versus classical CUSUM

The different *ARL*s values of the classical CUSUM chart proposed by Page[3] are given in Table II. The proposed MCE chart is performing really well for all values of λ_c. When $\lambda_c = 0.1$, the proposed chart has better ARL_1 values for small shifts (i.e., $\delta \leq 0.5$) as compared with the classical CUSUM. Similarly, at $\lambda_c = 0.25$, the proposed chart also has smaller ARL_1 values for $\delta \leq 0.75$. For all others values of λ_c, the proposed scheme is superior to the classical CUSUM for all possible shift amounts (cf. Table I vs. Table II). The EQL, RARL, and PCI of MCE are significantly smaller as compared with the classical CUSUM and for values of $\lambda_c \geq 0.5$ (cf. Table III).

3.4. Proposed mixed cumulative sum (CUSUM)–exponentially weighted moving average scheme versus runs rules CUSUM schemes

The runs rules-based CUSUM schemes are proposed by Riaz *et al.*[9] to enhance the performance of the CUSUM control charts . The ARL performance of their proposed schemes is provided in Table V where WL and AL represents the warning limits action limits, respectively. The proposed MCE chart is compared with their schemes at different values of λ_c. The proposed MCE scheme exhibits superior ARL performance relative to the runs rules-based CUSUM schemes for $\delta \leq 0.5$ shifts when $\lambda_c = 0.1$ and 0.25. However, for larger values of λ_c, the proposed MCE chart shows superiority for moderate and larger shifts. The EQL, RARL, and PCI measures also support the MCE scheme for relatively larger values of λ_c (cf. Table I vs Tables V and VI).

Table V. Average run length values of the runs rules cumulative sum (CUSUM) schemes

	Runs rules-based CUSUM scheme 1				Runs rules-based CUSUM scheme 2			
WL	**3.42**	**3.44**	**3.48**	**3.53**	**3.5**	**3.6**	**3.7**	**3.8**
AL	**4.8**	**4.6**	**4.4**	**4.2**	**4.44**	**4.19**	**4.08**	**4.03**
$\delta = 0$	168	168	168	168	168	168	168	168
0.25	71.87	72.26	71.94	71.40	71.49	72.94	73.11	73.59
0.5	25.56	25.65	25.59	25.30	25.36	25.37	25.37	25.40
0.75	13.54	13.50	13.50	13.33	13.40	13.35	13.31	13.28
1	8.66	8.57	8.52	8.40	8.46	8.38	8.34	8.32
1.5	5.08	5.01	4.94	4.83	4.94	4.83	4.78	4.75
2	3.68	3.61	3.52	3.42	3.54	3.42	3.38	3.35

Table VI. Performance measures of the proposed mixed cumulative sum (CUSUM)–exponentially weighted moving average and runs rules CUSUM

	Mixed CUSUM-EWMA				Runs rules CUSUM scheme $1k_z = 0.5$				Runs rules CUSUM scheme $2k_z = 0.5$			
					WL = 3.42	WL = 3.44	WL = 3.48	WL = 3.53	WL = 3.5	WL = 3.6	WL = 3.7	WL = 3.8
$ARL_0 = 168$	$\lambda_c = 0.1$	$\lambda_c = 0.25$	$\lambda_c = 0.5$	$\lambda_c = 0.75$	AL = 4.8	AL = 4.6	AL = 4.4	AL = 4.2	AL = 4.44	AL = 4.19	AL = 4.08	AL = 4.03
EQL	9.54	8.53	7.78	7.35	8.63	8.54	8.44	8.28	8.43	8.30	8.23	8.20
RARL	1.22	1.12	1.04	1.00	1.13	1.12	1.11	1.09	1.11	1.10	1.09	1.09
PCI	1.29	1.16	1.05	1.00	1.17	1.16	1.14	1.12	1.14	1.12	1.11	1.11

EQL, extra quadratic loss; RARL, relative average run length; PCI, performance comparison index.

3.5. Proposed mixed cumulative sum (CUSUM)–exponentially weighted moving average versus fast initial response CUSUM

The FIR CUSUM proposed by Lucas and Crosier[6] provides a head start to the CUSUM statistics. The ARLs of the CUSUM with FIR features are given in Table IV in which the head start is represented by C_0. The FIR features decreases the ARL_0 values of the CUSUM chart, and more importantly, this decreased ARL_0 becomes very small for large values of C_0 (e.g., for $C_0 = h/2$, the ARL_0 decreases from 168 to 149). Comparing with the MCE scheme at $\lambda_c = 0.1$ and 0.25 with the FIR CUSUM, we may see that the proposed MCE chart offers relatively better *ARL* properties, even if the FIR CUSUM is not having a fixed ARL_0. This implies that our proposed chart performs more efficiently (particularly at small and moderate shifts in comparison with the FIR CUSUM chart) (cf. Table I vs. Table IV).

3.6. Proposed mixed cumulative sum (CUSUM)–exponentially weighted moving average versus weighted CUSUM

Yashchin[8] proposed the weighted CUSUM in which he assigned weights (denoted by γ) to the past information in CUSUM statistics. The *ARL* values of the weighted CUSUM chart are given in Table VII. For $\delta \leq 1$, the proposed MCE chart is performing really better than the weighted CUSUM chart for the all values of λ_c (cf. Table I vs. Table VII).

3.7. Proposed mixed cumulative sum (CUSUM)–exponentially weighted moving average (EWMA) versus mixed EWMA-CUSUM

The MEC scheme is proposed by Abbas *et al.*,[13] and some results are given in Table II. At $\lambda_c = 0.1$ and $\lambda_c = 0.25$, the proposed MCE is compared with the MEC scheme, and it is found that MEC has superior performance with respect to the MCE scheme for $\delta \leq 0.5$, but if the value of δ increases (i.e., $\delta \geq 0.75$), the proposed MCE scheme offers relatively better performance with respect to the MEC scheme (cf. Table I vs. Table II).

In comparison with FIR EWMA (cf. Steiner[7]) and an adaptive CUSUM with EWMA-based shift estimator (cf. Jiang *et al.*[26]), the MEC scheme is preferable to our proposed MCE scheme because of its sensitivity for smaller shifts (cf. Abbas *et al.*[13]).

3.8. Proposed versus Shewhart schemes

In this subsection, the proposed scheme is compared with the classical Shewhart scheme. For the said scheme, ARL results are provided in Table II along with some other existing schemes. It is obvious that the proposed MCE scheme offers superior ability relative to the Shewhart scheme (cf. Table I versus Table II). We have also computed the standard deviation run length (SDRL) results for the proposed and Shewhart schemes (cf. Table VIII). It is observed that SDRL results of the proposed scheme remain smaller than those of the Shewhart scheme. A comparative graph of ARL and SDRL of both the schemes is provided in Figure 4 that highlights relative superiority of the proposed scheme over the Shewhart scheme in terms of ARL and SDRL.

It is interesting to note that the proposed MCE scheme is taking an edge over Shewhart scheme even for larger values of δ for varying choices of λ, which is not the case with MEC scheme. Moreover, the MEC scheme has relatively smaller SDRLs as compared with the MCE scheme for smaller values of δ and λ. The differences between SDRL values of MCE and MEC schemes keep decreasing with the increase in the values of δ and λ (which is mainly the dominance zone for the proposed scheme, cf. Tables I, II, and VIII).

3.9. Graphical presentation

We have provided some graphical presentations of *ARL* curves to show the superiority of our proposed MCE chart over others. For the sake of brevity, we have selected only four figures of different charts (discussed in Tables 1–7). In Figure 1, the proposed MCE scheme is compared with the classical EWMA and the runs rules-based EWMA. The *ARL* curve of the proposed MCE is at the lower side, which is evidence of superior performance of the proposed chart over EWMA and runs rules-based EWMA.

In Figure 2, the proposed MCE at $\lambda_c = 0.1$ is compared with the classical and weighted CUSUM and the MEC. The proposed MCE scheme has better performance as compared with the MEC scheme (for moderate to large shifts ($\delta \geq 0.5$) and for the weighted CUSUM at all shifts).

In Figure 3, the proposed MCE scheme is compared with the classical EWMA for two-sided shifts. Our scheme has superior performance over the classical EWMA for all shifts.

The proposed MCE scheme is also compared with the classical Shewhart scheme. Our scheme has superior performance over the classical Shewhart for all shifts even at small value of λ_c.

Table VII. Symmetric two-sided weighted cumulative sum scheme at $ARL_0 = 500$

δ	$h = 3.16$, $\gamma = 0.7$	$h = 3.46$, $\gamma = 0.8$	$h = 3.97$, $\gamma = 0.9$	$h = 5.09$, $\gamma = 1$
0	500	500	500	500
0.5	86.3	70.2	54.4	39
1	15.9	13.3	11.4	10.5
1.5	6.08	5.66	5.5	5.81
2	3.52	3.5	3.6	4.02

Table VIII. Standard deviation run length values of the proposed mixed cumulative sum (CUSUM)–exponentially weighted moving average (EWMA) (MCE), Shewhart, and mixed EWMA-CUSUM (MEC) schemes at $ARL_0 = 170$

	MCE				Shewhart	MEC			
	$\lambda_C = 0.1$	$\lambda_C = 0.25$	$\lambda_C = 0.5$	$\lambda_C = 0.75$		$\lambda_C = 0.1$	$\lambda_C = 0.25$	$\lambda_C = 0.5$	$\lambda_C = 0.75$
δ	$L_C = 5.96$	$L_C = 6.4$	$L_C = 5.4$	$L_C = 4.6$	$L = 2.753$	$h_Z = 21.3$	$h_Z = 13.29$	$h_Z = 8.12$	$h_Z = 5.48$
0	162.80	169.27	166.36	169.97	168.61	149.56	154.88	158.17	164.66
0.25	60.71	64.87	68.68	71.81	130.77	34.16	42.23	50.60	60.40
0.5	18.34	20.08	21.29	21.77	78.35	10.23	12.17	14.90	17.26
0.75	7.66	8.61	8.68	9.13	43.25	5.12	5.68	6.65	7.40
1	4.26	4.35	4.59	4.84	24.40	3.24	3.30	3.74	4.09
1.25	2.64	2.69	2.87	2.97	14.46	2.27	2.20	2.41	2.60
1.5	1.89	1.88	1.93	2.02	8.80	1.75	1.63	1.70	1.83
1.75	1.46	1.40	1.43	1.50	5.74	1.39	1.28	1.31	1.37
2	1.14	1.10	1.10	1.15	3.96	1.16	1.05	1.04	1.09

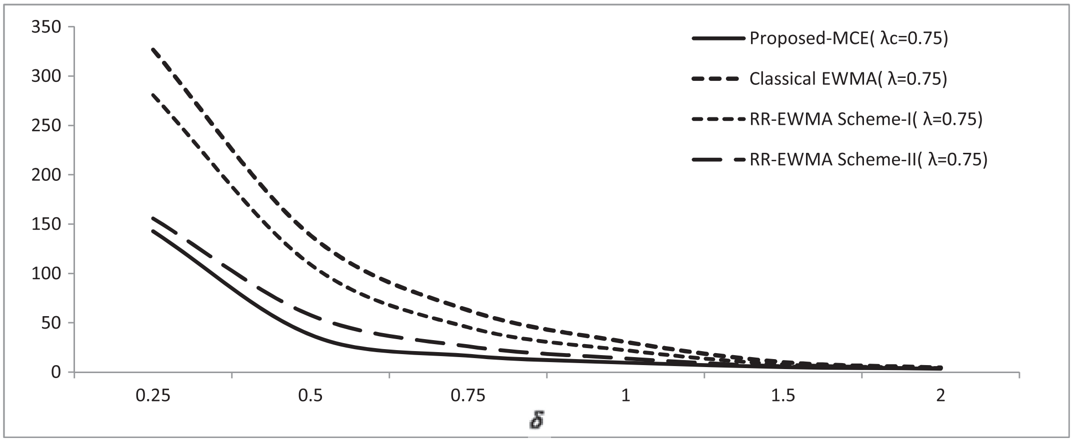

Figure 1. Average run length (ARL) comparisons of the proposed mixed cumulative sum–exponentially weighted moving average (MCE) scheme at $ARL_0 = 500$, $\lambda_c = 0.75$, and $k_z = 0.5$ with other competing charts

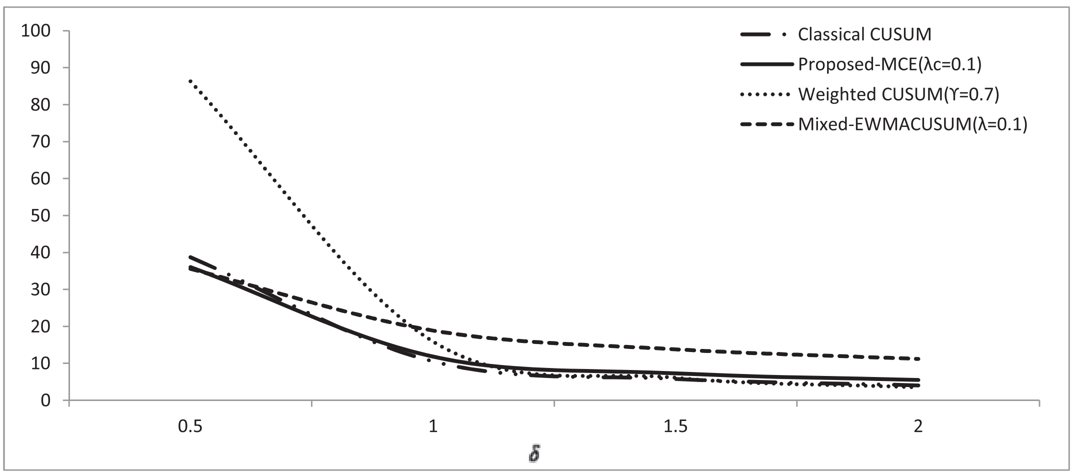

Figure 2. Average run length (ARL) comparisons of the proposed mixed cumulative sum (CUSUM)–exponentially weighted moving average (EWMA) (MCE) scheme at $ARL_0 = 500$, $\lambda_c = 0.1$, and $k_z = 0.5$ with the classical CUSUM, weighted CUSUM, and mixed EWMA-CUSUM schemes

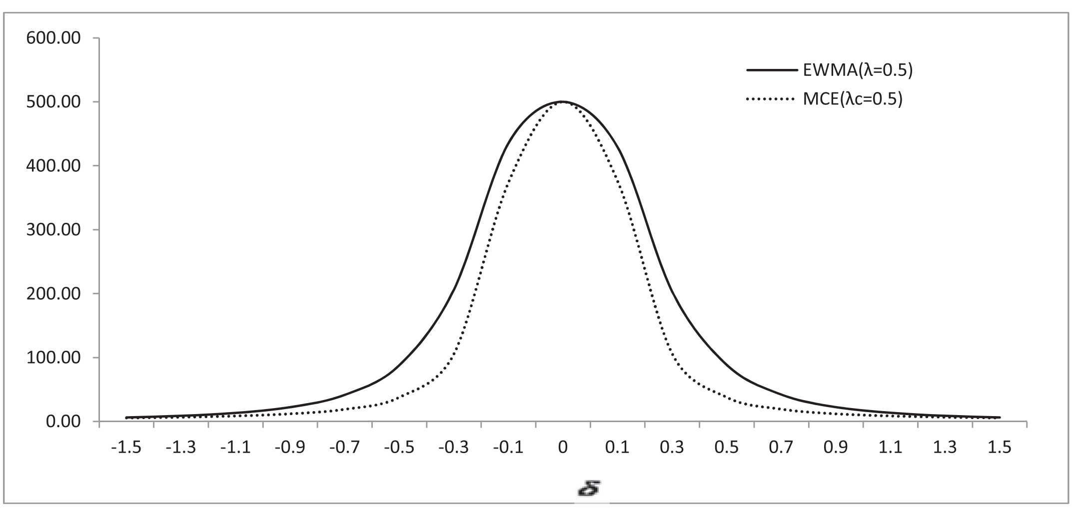

Figure 3. Average run length (ARL) comparison of the proposed mixed cumulative sum–exponentially weighted moving average (EWMA) (MCE) and classical EWMA at $ARL_0 = 500$

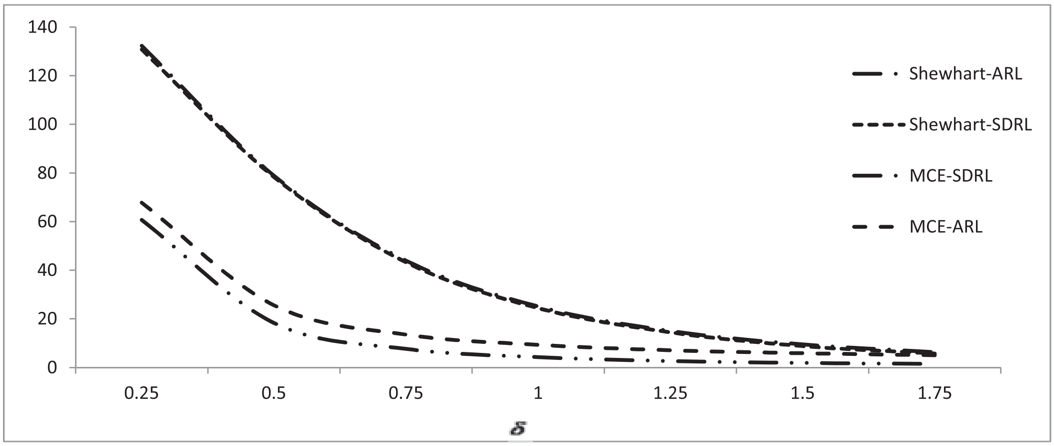

Figure 4. Average run length (ARL) and SDRL comparison of the proposed mixed cumulative sum–exponentially weighted moving average (MCE) and classical Shewhart at $ARL_0 = 200$

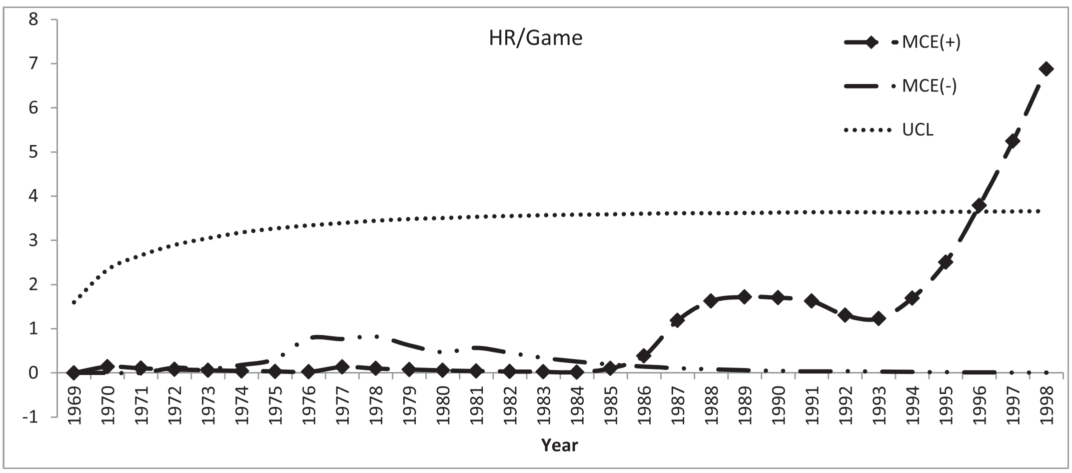

Figure 5. Control chart for the mixed cumulative sum–exponentially weighted moving average (MCE) scheme at $\lambda_c = 0.25$, $k_z = 0.5$, and $ARL_0 = 500$

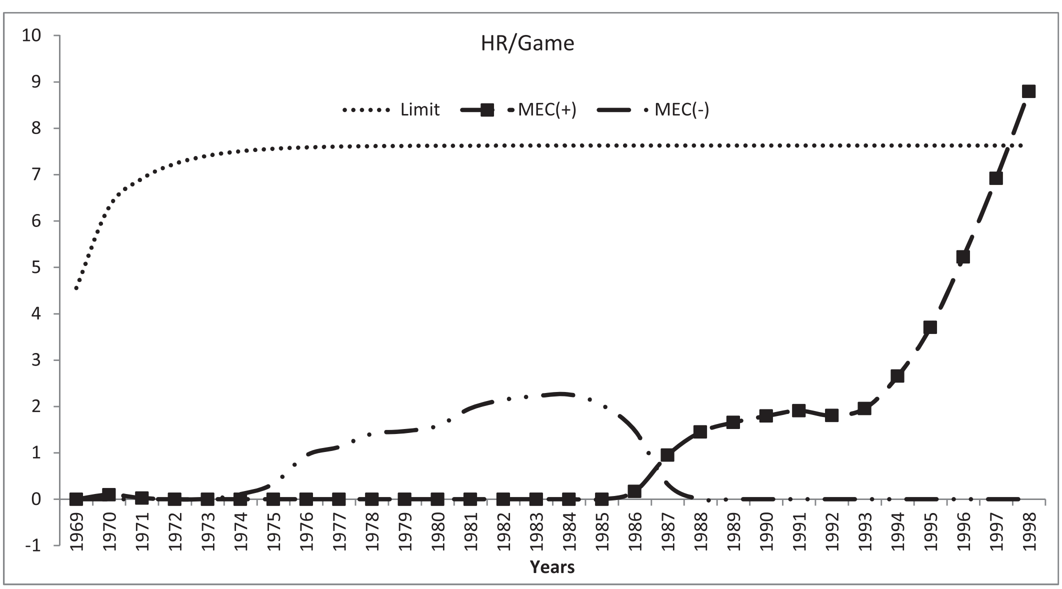

Figure 6. Control chart for mixed cumulative sum–exponentially weighted moving average (MEC) scheme at $\lambda_c = 0.25$, $k_z = 0.5$, and $ARL_0 = 500$

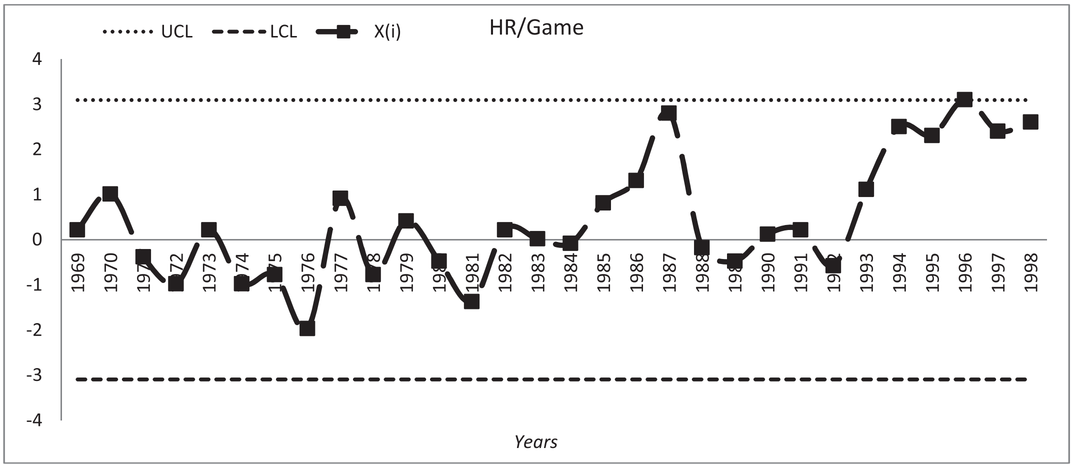

Figure 7. Control chart for the Shewhart scheme at $ARL_0 = 500$

4. Illustration of the proposed mixed cumulative sum–exponentially weighted moving average scheme with real datasets

To show the performance of the proposed MCE scheme with real data is important from a practical point of view. The Major League Baseball is the highest professional baseball league in the United States, and its modern history started in 1901. The Major League Baseball has experienced a series of historical eras since it begun; during the whole period, offensive performance was observed to be different than in other periods of baseball history. Hill and Schvaneveldt[27] used a data set from the period 1969 to 2008 to judge the offensive performance in baseball. One metric of offensive performance is measured as home run per game. In order to detect out of control points in the said dataset, we have used the proposed MCE scheme and received three out of control signals (cf. Figure 5). We have also considered two competing schemes from the two extreme ends, that is, Shewhart (where larger shifts are of major concern) and MEC (small and moderate shifts are of major concern). We have constructed the control charts (for the same data set) of two aforementioned competing schemes and received only one out of control signal for each scheme (cf. Figures 6, 7). The differences in the detection abilities of these three types of charts convey the message quite efficiently for our study purposes.

5. Conclusions and recommendations

This study has proposed a new control chart by combining the features of CUSUM and EWMA charts, called MCE control chart. The analysis has revealed that the proposed MCE control chart is very sensitive for the detection of small and moderate shifts and offers a quite efficient structure as compared with existing counterparts. The relative performance of the proposed chart as compared with the other charts varies depending on the amounts of shifts. The MCE scheme has superior performance as compared with an alternative MEC control chart (the MEC scheme) for $\delta \geq 0.5$ when $\lambda_c \geq 0.5$. Moreover, the proposed scheme is an efficient competitor to the usual Shewhart scheme for varying values of λ and δ.

For future research, the robustness of this chart can be checked with non-normal distributions in the comparison of other robust charting structures, and also, this study can be extended for monitoring the dispersion parameter, as well as for multivariate structure.

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References

1. Shewhart WA. Economic control of quality manufactured product. D. Van Nostrand. (reprinted by the American Society for Quality Control in 1980, Milwaukee, WI): New York, 1931.
2. Roberts SW. Control chart tests based on geometric moving averages. *Technometrics* 1959; **1**(3):239–250.
3. Page ES. Cumulative sum charts. *Technometrics* 1951; **3**(1):1–9.
4. Lucas JM. Combined Shewhart-CUSUM quality control schemes. *Journal of Quality Technology* 1982; **14**(2):51–59.
5. Lucas JM, Saccucci MS. Exponentially weighted moving average control schemes: properties and enhancements. *Technometrics* 1990; **32**(1):1–12.

6. Lucas J, Crosier R. Fast initial response for CUSUM quality-control scheme. *Technometrics* 1982; **24**:199–205.
7. Steiner S. EWMA control charts with time varying control limits and fast initial response. *Journal of Quality Technology* 1999; **31**(1):75–86.
8. Yashchin E. Weighted cumulative sum technique. *Technometrics* 1989; **31**(1):321–338.
9. Riaz M, Abbas N, Does RJMM. Improving the performance of CUSUM charts. *Quality and Reliability Engineering International* 2011; **27**(4):415–424.
10. Riaz M, Mehmood R, Does RJMM. On the performance of different control charting rules. *Quality and Reliability Engineering International* 2011; **27**(8):1059–1067.
11. Mehmood R, Riaz M, Does RJMM. Control charts for location based on different sampling schemes. *Journal of Applied Statistics* 2013; **40**(3):483–494.
12. Abbas N, Riaz M, Does RJMM. Enhancing the performance of EWMA charts. *Quality and Reliability Engineering International* 2011; **27**(6):821–833.
13. Abbas N, Riaz M, Does RJMM. Mixed exponentially weighted moving average–cumulative sum charts for process monitoring. *Quality and Reliability Engineering International* 2012; **29**(3):345–356.
14. Wu Z, Jiao JX, Yang M, Liu Y, Wang ZJ. An enhanced adaptive CUSUM control chart. *IIE Transactions* 2009; **41**(7):642–653.
15. Ryu JH, Wan H, Kim S. Optimal design of a CUSUM chart for a mean shift of unknown size. *Journal of Quality Technology* 2010; **42**(3):311–326.
16. Zhao Y, Tsun F, Wang Z. Dual CUSUM control schemes for detecting a range of mean shifts. *IIE Transactions* 2005; **37**(11):1047–1057.
17. Han D, Tsung F, Li Y. A CUSUM chart with local signal amplification for detecting a range of unknown shifts. *International Journal of Reliability, Quality and Safety Engineering* 2007; **14**(2):81–97.
18. Ou YJ, Wu Z, Tsung F. A comparison study of effectiveness and robustness of control charts for monitoring process mean. *International Journal of Production Economics* 2012; **135**(1):479–490.
19. Zhang S, Wu Z. Monitoring the process mean and variance by the WLC scheme with variable sampling intervals. *IIE Transactions* 2006; **38**(4):377–387.
20. Ou YJ, Wu Z, Goh TN. A new SPRT chart for monitoring process mean and variance. *International Journal of Production Economics* 2011; **132**(2):303–314.
21. Ahmad S, Riaz M, Abbasi SA, Lin Z. On efficient median control charting. *Journal of the Chinese Institute of Engineers* 2012: ; **37**(3):358–375.
22. Ahmad S, Riaz M, Abbasi SA, Lin Z. On median control charting under double sampling scheme. *European Journal of Industrial Engineering* 2012: in press.
23. Ahmad S, Riaz M, Abbasi SA, Lin Z. On monitoring process variability under double sampling scheme. *International Journal of Production Economics* 2013; **142**(2):388–400.
24. Hawkins DM. Cumulative sum control charting: an underutilized SPC tool. *Quality Engineering* 1993; **5**(3):463–477.
25. Montgomery DC. Introduction to statistical quality control (6th edn). Wiley: New York, 2009.
26. Jiang W, Shu L, Aplet D. Adaptive CUSUM procedures with EWMA-based shift estimators. *IIE Transactions* 2008; **40**(10):992–1003.
27. Hill SE, Schvaneveldt SJ. Using statistical process control charts to identify the steroids era in Major League Baseball: an educational exercise. *Journal of Statistics Education* 2011; 19.

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