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Memory-Type Control Charts for Monitoring the Process Dispersion

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Control charts have been broadly used for monitoring the process mean and dispersion. Cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts are memory control charts as they utilize the past information in setting up the control structure. This makes CUSUM and EWMA-type charts good at detecting small disturbances in the process. This article proposes two new memory control charts for monitoring process dispersion, named as floating $T-S^2$ and floating $U-S^2$ control charts, respectively. The average run length (*ARL*) performance of the proposed charts is evaluated through a simulation study and is also compared with the CUSUM and EWMA charts for process dispersion. It is found that the proposed charts are better in detecting both positive as well as negative shifts. An additional comparison shows that the floating $U-S^2$ chart has slightly smaller *ARL*s for larger shifts, while for smaller shifts, the floating $T-S^2$ chart has better performance. An example is also provided which shows the application of the proposed charts on simulated datasets.

Keywords: average run length; control chart; Johnson S_B transformation; logarithmic transformation; process variability; statistical process control (SPC)

1. Introduction

Variations in a manufacturing process can be categorized into common cause and special cause variations. In the presence of common cause variation only, a process is considered in-control, but once special cause variations sum up with the common cause variations, the process is stated out-of-control. Control charts are very popular due to their capability to detect the presence of special cause variations and hence to operationalize whether a process is out-of-control or not. The presence of special cause variation is generally limited on the location or/and spread parameters of the process. A process can go from in-control to out-of-control situation if the mean of that process is shifted to a new level. Similarly, an increased spread will also cause inconsistency in the process resulting into an out-of-control situation. In contrast, any decrease in the spread parameter may improve the quality of that process (see Montgomery[1] for more details). The present article only deals with monitoring the spread/dispersion parameter of a process.

Shewhart[2] started the concept of control charts with some useful charts for monitoring the process dispersion, e.g. the range (*R*), the standard deviation (*S*), and the variance (S^2) charts. The drawback of these charts is that their interpretation is merely based on the present sample which means that they pay no attention to the past data resulting into a relatively bad performance for small disturbances in the process. This deficiency of Shewhart-type charts is covered by the memory-type control charts which achieve better performance for detecting small shifts as they exploit past data along with present data. Popular memory-type control structures are the cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts. There is a lot of literature available on CUSUM and EWMA-type control charts for monitoring the process dispersion, e.g. see Page,[3] Hawkins,[4] Acosta-Mejia et al.[5] and Chang and Gan[6] for CUSUM-type charts and Ng and Case,[7] Crowder and Hamilton[8] and Huwang et al.[9] for EWMA-type charts.

Most of these charts are based on transforming the sample variance such that the new transformed statistic may be closely approximated by a normally distributed variable and hence applying the usual CUSUM and EWMA structures (recommended by Page[10] and Roberts,[11] respectively) on it. In a similar direction, Castagliola[12] proposed a new EWMA $\ln-S^2$ chart for monitoring the process dispersion. He used a logarithmic three-parameter transformation to obtain a normal approximation for the sample variance. A similar transformation is used by Castagliola et al.[13] to set up a CUSUM S^2 chart for monitoring process dispersion. Following their previous work, Castagliola et al.[14] proposed an EWMA $J-S^2$ chart based on a four parameter Johnson transformation.

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Recently, Abbas et al.[15] proposed a new memory-type control chart for location, which they named as progressive mean control chart. They show that the progressive mean chart is better than the standard EWMA and CUSUM control charts and some of their modifications. An extension to their work in a non-parametric setting is proposed by Abbasi et al..[16] Following the structure of progressive charts, we propose two new memory-type control charts for monitoring the process dispersion, named as floating $T-S^2$ chart (which is based on a three-parameter logarithmic transformation) and floating $U-S^2$ chart (based on a four parameter Johnson transformation). The average run length (*ARL*) is used as the performance measure which is defined as the average number of subgroups that should be monitored before an out-of-control signal is received. ARL_0 is referred as the in-control *ARL*, and ARL_1 is the notation used for the out-of-control *ARL*.

In the next section, the details regarding the proposed charts are provided. Comparison of the proposed chart with some of the memory-type control charts for monitoring the process dispersion is given in Section 3. Section 4 contains the implementation of the proposed chart on a simulated dataset, whereas the summary and conclusions are given in Section 5.

2. The proposed floating control charts

Let $X_{j,1}, X_{j,2}, \ldots, X_{j,i}, \ldots, X_{j,n}$ be a random sample from a normal distribution with mean μ and variance σ_0^2, i.e.

$$X_{j,i} \sim N\left(\mu, \sigma_0^2\right) \text{ for all } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots \ldots \tag{1}$$

Let $S_j^2 = \sum_{i=1}^{n} \left(X_{j,i} - \bar{X}_j\right)^2/(n-1)$ be the sample variance of the j^{th} sample. Under (1), it is known through the probability distribution theory that S_j^2 follows a chi-square distribution, i.e. $S_j^2 \sim \frac{\sigma_0^2}{n-1}\chi^2_{(n-1)}$. Castagliola[12] discussed that if we transform S_j^2 using a three-parameter logarithmic transformation, the resulting transformed variable (denoted by T_j) approximately follows a normal distribution with mean $\mu_T(n)$ and variance $\sigma_T^2(n)$. Hence, from Castagliola[12], we obtain that

$$T_j = a_T + b_T \ln\left(S_j^2 + c_T\right) \tag{2}$$

where $b_T = B_T(n)$, $c_T = C_T(n)\sigma_0^2$ and $a_T = A_T(n) - 2B_T(n) ln(\sigma_0)$. Table I provides the values of $\mu_T(n), \sigma_T^2(n), A_T(n), B_T(n)$ and $C_T(n)$ for $n = 3, 4, 5, \ldots\ldots\ldots, 15$. For more details on the distribution of T_j and calculation of the constants, see Castagliola.[12]

Castagliola et al.[14] proposed another similar type of transformation based on a four parameter Johnson S_B transformation. They claimed that this four parameter transformation gives a better approximation to the normal distribution as compared to the three-parameter logarithmic transformation. With the notation of Castagliola et al.,[14] it follows that

$$U_j = a_U + b_U \ln\left(\frac{S_j^2 - c_U}{d_U + c_U - S_j^2}\right) \tag{3}$$

where $a_U = A_U(n), b_U = B_U(n), c_U = C_U(n)\sigma_0^2$ and $d_U = D_U(n)\sigma_0^2$. Variable U_j in (3) follows approximately a normal distribution with mean $\mu_U(n)$ and variance $\sigma_U^2(n)$ where the values of $\mu_U(n), \sigma_U^2(n), A_U(n), B_U(n), C_U(n)$ and $D_U(n)$ for $n = 3, 4, 5, \ldots, 15$ are given in Table II.

Note that in case of $d_U + c_U - S_j^2 \leq 0$, the transformation given in (3) is not possible, but Castagliola et al.[14] showed that the probability of occurrence of this event is so close to zero that it can be neglected. From the values of c_U and d_U, it can be noticed that

Table I. Values of $\mu_T(n), \sigma_T(n), A_T(n), B_T(n)$ and $C_T(n)$

n	$\mu_T(n)$	$\sigma_T(n)$	$A_T(n)$	$B_T(n)$	$C_T(n)$
3	0.02472	0.9165	−0.6627	1.8136	0.6777
4	0.01266	0.9502	−0.7882	2.1089	0.6261
5	0.00748	0.9670	−0.8969	2.3647	0.5979
6	0.00485	0.9765	−0.9940	2.5941	0.5801
7	0.00335	0.9825	−1.0827	2.8042	0.5678
8	0.00243	0.9864	−1.1647	2.9992	0.5588
9	0.00182	0.9892	−1.2413	3.1820	0.5519
10	0.00141	0.9912	−1.3135	3.3548	0.5465
11	0.00112	0.9927	−1.3820	3.5189	0.5421
12	0.00090	0.9938	−1.4473	3.6757	0.5384
13	0.00074	0.9947	−1.5097	3.8260	0.5354
14	0.00062	0.9955	−1.5697	3.9705	0.5327
15	0.00052	0.9960	−1.6275	4.1100	0.5305

Table II. Values of $\mu_U(n), \sigma_U(n), A_U(n), B_U(n)$ and $C_U(n)$

n	$\mu_U(n)$	$\sigma_U(n)$	$A_U(n)$	$B_U(n)$	$C_U(n)$	$D_U(n)$
3	0.0184	0.9475	3.1936	1.1952	−0.2588	15.077
4	0.0078	0.9739	3.3657	1.3983	−0.2438	12.591
5	0.0039	0.9852	3.5402	1.5727	−0.2352	11.312
6	0.0022	0.9908	3.7111	1.7281	−0.2295	10.530
7	0.0014	0.994	3.8768	1.8698	−0.2254	10.000
8	0.0009	0.9958	4.0369	2.0010	−0.2224	9.618
9	0.0006	0.9970	4.1918	2.1238	−0.2200	9.328
10	0.0004	0.9978	4.3417	2.2396	−0.2181	9.100
11	0.0003	0.9983	4.4869	2.3495	−0.2166	8.917
12	0.0002	0.9987	4.6279	2.4544	−0.2152	8.766
13	0.0002	0.9989	4.7648	2.5549	−0.2141	8.640
14	0.0001	0.9991	4.8981	2.6515	−0.2132	8.532
15	0.0001	0.9993	5.0279	2.7446	−0.2123	8.440

$d_U + c_U - S_j^2 \leq 0$ implies a very large value of S_j^2 as compared to the value of σ_0^2 so it can be taken as an out-of-control situation with a large positive shift. For more about the distributional properties of U_j, see Castagliola et al..[14]

Furthermore, it should be noted that any change in the process standard deviation will change the mean of the normalized variables given in (2) and (3). Hence, based on the these two (approximately) normalized statistics, we are now able to define our new control structures, denoted as floating $T-S^2$ and floating $U-S^2$ charts, respectively. These charts monitor basically the mean of the transformed statistics in (2) and (3) and hence control the process dispersion.

2.1. *Floating* $T-S^2$ control chart

The first proposed chart, named as floating $T-S^2$ chart, is based on the three-parameter logarithmic transformation given in (2). The plotting statistic is given as:

$$FT_j = \frac{\sum_{k=1}^{j} T_k}{j} \tag{4}$$

The statistic in (4) is a cumulative average of the three-parameter logarithmic transformation given in (2). According to the probability distribution theory, we have that, if T_j follows (approximately) a normal distribution with mean $\mu_T(n)$ and variance $\sigma_T^2(n)$, then $FT_j = \sum_{k=1}^{j} T_k/j$ will also be approximately normal (for a fixed value of j) with mean $\mu_T(n)$ and variance $\frac{\sigma_T^2(n)}{j}$. This implies that the control limits (including the upper control limit (UCL), center line (CL) and lower control limit (LCL)) for the floating statistic given in (4) can be defined as:

$$LCL_j = \mu_T(n) - K_T\frac{\sigma_T(n)}{\sqrt{j}},\ CL = \mu_T(n),\ UCL_j = \mu_T(n) + K_T\frac{\sigma_T(n)}{\sqrt{j}} \tag{5}$$

where the width of the control limits is determined by K_T. The ARL_0 can be controlled by adjusting this constant (K_T) as the $ARLs$ for a control chart with wider limits are larger and vice versa. A problem seen in the above structure is that, once the value of j becomes larger, it becomes almost impossible for the plotting statistic in (4) to cross the control limits in (5), in case of shifted variance. This implies that the width of the control structure in (5) remains too wide for the larger values of j (wide relative to the plotting statistic). Note that if a process shift has occurred from time zero, then using the progressive mean, i.e. floating $T-S^2$ chart from (4) averages all observed process results with equal weight, and all of them reflect the process change. However, if the process shift occurs at some other time, the progressive mean chart will not be so good. Now, it gives equal weight to some observations before the process change and thus underestimates/overestimates the current process variance. This issue is resolved by putting a function of j, i. e. $f(j) = j^q$, in the denominator of $\sigma_T(n)$ such that the limits become a bit narrower for the larger values of j (cf. Abbas et al.,[15] where the same kind of penalty function is used). This results in control limits for the proposed floating $T-S^2$ chart as:

$$LCL_{Tj} = \mu_T(n) - K_T'\frac{\sigma_T(n)}{j^{q+0.5}},\ CL_T = \mu_T(n), UCL_{Tj} = \mu_T(n) + K_T'\frac{\sigma_T(n)}{j^{q+0.5}} \tag{6}$$

where K_T' is the adjusted control limit coefficient, and it becomes the design parameter for the proposed chart along with q. Note that the control limits given in (5) are a special case of the limits in (6) with $q=0$. Tables III–VII contain the ARL values for the proposed floating $T-S^2$ chart where $\delta = \sigma_1/\sigma_0$ represents the amount of shift in the process standard deviation. Standard deviation of run

Table III. ARL (*SDRL* in parentheses) values of floating $T-S^2$ chart with $n=3$ and $ARL_0 \cong 370$

δ	$q=0.15$	$q=0.2$	$q=0.25$	$q=0.3$	$q=0.35$	$q=0.4$
	$K_T' = 2.724$	$K_T' = 3.568$	$K_T' = 4.68$	$K_T' = 6.152$	$K_T' = 8.1$	$K_T' = 10.67$
0.5	5.61 (1.7)	7.16 (1.79)	8.91 (1.89)	10.84 (1.96)	12.94 (2.03)	15.18 (2.09)
0.6	7.64 (3.05)	9.58 (3.22)	11.77 (3.38)	14.11 (3.49)	16.6 (3.58)	19.23 (3.67)
0.7	11.52 (5.95)	14.17 (6.22)	16.96 (6.42)	19.99 (6.62)	23.13 (6.7)	26.32 (6.81)
0.8	20.4 (13.68)	24.6 (14.16)	28.65 (14.49)	32.94 (14.68)	37.08 (14.72)	41.19 (14.8)
0.9	52.35 (48.55)	61.12 (49.51)	69.13 (49.86)	76.05 (49.72)	82.5 (49.41)	88.32 (48.6)
0.95	115.8 (143.2)	134.7 (143.4)	150.1 (143.6)	161.9 (140.8)	171.5 (137.2)	179.2 (134.9)
1	371.5 (1035.6)	368.9 (736.9)	370.4 (570.9)	371.3 (476.9)	370.7 (411.1)	369.0 (366.1)
1.05	109.2 (147.02)	129.3 (150.5)	145.7 (150.7)	158.5 (148.6)	168.8 (145.1)	177.7 (141.7)
1.1	50.28 (55.64)	60.35 (58.07)	69.57 (59.71)	77.51 (60.14)	84.68 (59.47)	90.91 (59.08)
1.2	20.99 (19.99)	25.75 (21.13)	30.17 (21.82)	34.81 (22.35)	39.2 (22.5)	43.67 (22.6)
1.3	12.3 (10.87)	15.43 (11.8)	18.49 (12.2)	21.77 (12.65)	25.15 (12.99)	28.41 (13.03)
1.4	8.55 (7.26)	10.78 (7.84)	13.14 (8.27)	15.63 (8.54)	18.36 (8.83)	21.15 (9.08)
1.5	6.52 (5.33)	8.21 (5.76)	10.15 (6.15)	12.24 (6.42)	14.51 (6.65)	16.9 (6.86)
2	3 (2.17)	3.8 (2.42)	4.82 (2.63)	5.96 (2.8)	7.27 (2.97)	8.71 (3.1)
3	1.69 (1.01)	2.03 (1.18)	2.56 (1.3)	3.24 (1.37)	4.02 (1.49)	4.91 (1.57)

Table IV. ARL (*SDRL* in parentheses) values of floating $T-S^2$ chart with $n=5$ and $ARL_0 \cong 370$

δ	$q=0.15$	$q=0.2$	$q=0.25$ b	$q=0.3$	$q=0.35$	$q=0.4$
	$K_T' = 2.724$	$K_T' = 3.568$	$K_T' = 4.68$	$K_T' = 6.152$	$K_T' = 8.1$	$K_T' = 10.67$
0.5	3.4 (0.95)	4.43 (1.01)	5.63 (1.07)	7.01 (1.13)	8.55 (1.17)	10.22 (1.22)
0.6	4.58 (1.67)	5.9 (1.78)	7.39 (1.89)	9.09 (1.99)	10.93 (2.04)	12.92 (2.12)
0.7	6.85 (3.21)	8.66 (3.42)	10.68 (3.58)	12.85 (3.72)	15.19 (3.83)	17.69 (3.94)
0.8	12.24 (7.43)	15.03 (7.78)	18.02 (8.1)	21.12 (8.23)	24.34 (8.38)	27.68 (8.49)
0.9	32.09 (26.71)	38.39 (27.69)	44.05 (28.26)	49.23 (28.15)	54.54 (28.13)	59.67 (28.06)
0.95	77.92 (85.43)	91.23 (86.51)	101.7 (85.95)	111.4 (86.1)	119.2 (84.48)	126.7 (83.39)
1	369.1 (1019.9)	368.9 (734.8)	370.9 (570.2)	370.6 (474.6)	370.2 (412.4)	369.5 (366.0)
1.05	74.34 (87.98)	89.12 (91.88)	101.1 (92.03)	111.0 (92.83)	120.3 (92.6)	127.3 (90.36)
1.1	31.95 (31.47)	38.79 (32.97)	45.02 (33.92)	50.88 (34.34)	56.34 (34.45)	61.32 (33.95)
1.2	12.92 (10.85)	16.09 (11.55)	19.26 (12.05)	22.51 (12.34)	25.91 (12.62)	29.38 (12.76)
1.3	7.66 (5.94)	9.64 (6.37)	11.8 (6.71)	14.16 (7.01)	16.68 (7.22)	19.26 (7.35)
1.4	5.35 (3.9)	6.79 (4.21)	8.42 (4.49)	10.25 (4.73)	12.26 (4.92)	14.35 (5.08)
1.5	4.09 (2.84)	5.23 (3.14)	6.56 (3.36)	8.05 (3.53)	9.71 (3.71)	11.51 (3.84)
2	2 (1.17)	2.52 (1.33)	3.21 (1.43)	4.03 (1.54)	4.99 (1.64)	6.07 (1.72)
3	1.25 (0.51)	1.43 (0.64)	1.78 (0.76)	2.3 (0.77)	2.85 (0.84)	3.52 (0.88)

lengths (*SDRL*) are given in the parentheses. Due to a number of difficulties faced in applying the Markov chain procedure for approximating the run length properties, these properties are evaluated by running 10^5 simulations. The main reason, that we were not able to generalize the procedure of Steiner,[17] is that the distribution of FT_j statistic keeps varying with j, which disturbs the partitioning of the states at every time point j. It makes the computation of transition probabilities quite cumbersome for the floating chart. This is not the case with the EWMA chart (cf. Steiner).[17] The simulation program is developed in R language and available on request from the authors.

Tables III-VI indicate that an increase in the value of q increases the ARL_1 values and decreases the value of $SDRL_0$ for a fixed value of n. In general, a large value of $SDRL_0$ is not recommended for a control structure (cf. Ryan,[18] and Govindaraju and Zhang[19]), and also smaller values of ARL_1 are desired so that a shift is detected as early as possible. Therefore, it becomes a tradeoff between ARL_1 values and $SDRL_0$ by adjusting the design parameter q.

Also, we note that as the value of q approaches 0, the value of $SDRL_0$ tends to infinity which makes it impossible to compute the ARL_0 value.

2.2. *Floating* $U-S^2$ control chart

The plotting statistic for the second proposed chart (based on a four parameter Johnson S_B transformation) to monitor the process dispersion is given as:

Table V. *ARL* (*SDRL* in parentheses) values of floating $T-S^2$ chart with $n=7$ and $ARL_0 \cong 370$

δ	$q=0.15$	$q=0.2$	$q=0.25$	$q=0.3$	$q=0.35$	$q=0.4$
	$K_T' = 2.724$	$K_T' = 3.568$	$K_T' = 4.68$	$K_T' = 6.152$	$K_T' = 8.1$	$K_T' = 10.67$
0.5	2.59 (0.67)	3.39 (0.73)	4.35 (0.79)	5.46 (0.83)	6.73 (0.86)	8.14 (0.9)
0.6	3.44 (1.18)	4.48 (1.27)	5.68 (1.36)	7.05 (1.42)	8.58 (1.49)	10.27 (1.56)
0.7	5.13 (2.27)	6.54 (2.42)	8.15 (2.55)	9.95 (2.67)	11.92 (2.77)	14.03 (2.85)
0.8	9.08 (5.19)	11.31 (5.46)	13.75 (5.69)	16.36 (5.89)	19.11 (6.05)	22.01 (6.14)
0.9	24.19 (18.95)	29.02 (19.67)	33.68 (19.93)	38.15 (20.17)	42.77 (20.28)	47.25 (20.22)
0.95	60.22 (61.4)	71.02 (62.63)	79.77 (63.02)	87.60 (62.78)	95.06 (62.72)	100.9 (61.39)
1	368.9 (1021.6)	368.9 (737.46)	370.5 (571.14)	371.2 (478.75)	371.8 (412.52)	371.7 (366.86)
1.05	58.25 (64.78)	69.91 (67.04)	79.80 (68.2)	88.46 (68.59)	95.70 (67.68)	102.7 (67.29)
1.1	24.28 (22.33)	29.61 (23.48)	34.75 (24.21)	39.4 (24.36)	44.41 (24.87)	48.9 (24.62)
1.2	9.75 (7.57)	12.23 (8.12)	14.77 (8.49)	17.61 (8.86)	20.4 (9.05)	23.4 (9.16)
1.3	5.77 (4.1)	7.38 (4.49)	9.1 (4.73)	11.04 (4.97)	13.14 (5.15)	15.38 (5.28)
1.4	4.08 (2.72)	5.22 (2.96)	6.53 (3.17)	8.04 (3.36)	9.68 (3.5)	11.5 (3.66)
1.5	3.15 (1.98)	4.03 (2.17)	5.11 (2.34)	6.34 (2.51)	7.72 (2.64)	9.24 (2.75)
2	1.6 (0.81)	2 (0.95)	2.56 (1.02)	3.24 (1.11)	4.02 (1.17)	4.93 (1.24)
3	1.1 (0.32)	1.21 (0.44)	1.45 (0.57)	1.89 (0.6)	2.37 (0.58)	2.92 (0.67)

Table VI. *ARL* (*SDRL* in parentheses) values of floating $T-S^2$ chart with $n=9$ and $ARL_0 \cong 370$

δ	$q=0.15$	$q=0.2$	$q=0.25$	$q=0.3$	$q=0.35$	$q=0.4$
	$K_T' = 2.724$	$K_T' = 3.568$	$K_T' = 4.68$	$K_T' = 6.152$	$K_T' = 8.1$	$K_T' = 10.67$
0.5	2.18 (0.51)	2.81 (0.63)	3.63 (0.65)	4.6 (0.67)	5.71 (0.71)	6.95 (0.74)
0.6	2.85 (0.91)	3.70 (1.01)	4.72 (1.08)	5.92 (1.14)	7.26 (1.20)	8.74 (1.25)
0.7	4.18 (1.75)	5.38 (1.89)	6.76 (2.00)	8.33 (2.12)	10.05 (2.2)	11.93 (2.28)
0.8	7.38 (4.02)	9.29 (4.29)	11.36 (4.48)	13.63 (4.66)	16.11 (4.79)	18.64 (4.9)
0.9	19.70 (14.78)	23.81 (15.42)	27.83 (15.65)	32.01 (15.93)	36.07 (16.05)	40.23 (16.11)
0.95	49.78 (48.29)	59.26 (49.96)	66.86 (50.77)	73.83 (50.29)	80.36 (49.83)	86.3 (49.4)
1	368.1 (1027.8)	369.9 (737.0)	371.3 (574.9)	371.4 (479.1)	369.3 (409.4)	368.2 (363.9)
1.05	48.61 (51.39)	58.69 (53.49)	67.36 (54.44)	74.35 (54.37)	81.35 (54.35)	87.41 (53.88)
1.1	20.03 (17.46)	24.52 (18.43)	28.77 (18.85)	33.09 (19.34)	37.42 (19.55)	41.65 (19.68)
1.2	7.99 (5.89)	10.03 (6.31)	12.31 (6.69)	14.68 (6.91)	17.25 (7.13)	20.0 (7.34)
1.3	4.76 (3.17)	6.08 (3.47)	7.6 (3.7)	9.29 (3.89)	11.13 (4.06)	13.13 (4.2)
1.4	3.39 (2.10)	4.33 (2.30)	5.5 (2.49)	6.77 (2.64)	8.24 (2.78)	9.84 (2.89)
1.5	2.63 (1.54)	3.38 (1.72)	4.3 (1.84)	5.36 (1.97)	6.58 (2.09)	7.92 (2.18)
2	1.4 (0.63)	1.71 (0.76)	2.19 (0.83)	2.78 (0.87)	3.46 (0.94)	4.27 (0.99)
3	1.04 (0.2)	1.1 (0.31)	1.27 (0.46)	1.64 (0.55)	2.11 (0.45)	2.55 (0.57)

$$FU_j = \frac{\sum_{k=1}^{j} U_k}{j} \tag{7}$$

Like FT_j in (4), here FU_j also follows approximately a normal distribution with mean $\mu_U(n)$ and variance $\frac{\sigma_U^2(n)}{j}$. Therefore, the control limits for this second proposed chart, named as floating $U-S^2$ chart, are given as:

$$LCL_{Uj} = \mu_U(n) - K_U' \frac{\sigma_U(n)}{j^{q+0.5}}, \; CL_U = \mu_U(n), UCL_{Uj} = \mu_U(n) + K_U' \frac{\sigma_U(n)}{j^{q+0.5}} \tag{8}$$

where K_U' is the control limit coefficient for this second proposed chart. The *ARL* and *SDRL* values for the floating $U-S^2$ chart are given in Table VII with $q=0.3$, $K_U' = 6.152$ and $ARL_0 \cong 370$. For other values of q, similar results can easily be obtained.

From Tables III-VII, we may conclude that:

i. both floating charts are performing good, not only for positive shifts but also for negative shifts in the process standard deviation;

Table VII. *ARL* (*SDRL* in parentheses) values of floating $U-S^2$ chart with $q=0.3$, $K_T'=6.152$ and $ARL_0 \cong 370$

δ	$n=3$	$n=5$	$n=7$	$n=9$
0.5	10.34 (2.15)	6.62 (1.22)	5.14 (0.89)	4.33 (0.73)
0.6	13.76 (3.76)	8.77 (2.10)	6.79 (1.51)	5.69 (1.21)
0.7	19.86 (7.02)	12.66 (3.91)	9.77 (2.78)	8.15 (2.18)
0.8	33.18 (15.44)	21.12 (8.53)	16.26 (6.07)	13.53 (4.77)
0.9	77.39 (51.33)	49.66 (28.91)	38.26 (20.55)	32.01 (16.21)
0.95	164.07 (143.25)	112.22 (87.55)	88.15 (63.55)	74.04 (50.63)
1	371.22 (482.6)	370.25 (477.14)	371.26 (479.81)	369.62 (480.89)
1.05	161.35 (150.65)	112.68 (93.37)	89.01 (68.96)	74.84 (54.97)
1.1	78.84 (60.92)	51.46 (34.58)	39.89 (24.61)	33.41 (19.52)
1.2	35.56 (22.69)	22.92 (12.58)	17.64 (8.82)	14.75 (6.97)
1.3	22.32 (12.92)	14.31 (7.03)	11.10 (4.99)	9.29 (3.92)
1.4	15.98 (8.73)	10.35 (4.82)	8.05 (3.42)	6.75 (2.68)
1.5	12.41 (6.61)	8.06 (3.62)	6.29 (2.57)	5.28 (2.03)
2	5.74 (2.97)	3.78 (1.67)	3.00 (1.22)	2.55 (0.98)
3	2.76 (1.54)	1.88 (0.89)	1.53 (0.65)	1.34 (0.52)

ii. for a fixed ARL_0, the proposed floating $T-S^2$ chart is performing better for small shifts, like $\delta \in [0.9,1.3]$, whereas the performance of floating $U-S^2$ chart is better for large shifts, like $\delta \le 0.8$ and $\delta \ge 1.4$;
iii. for fixed values of q and ARL_0, the values of the control limit coefficients are the same for both proposed charts;
iv. for larger values of n, the *ARL* values for both charts are more symmetric with respect to δ as the distribution of both T_j and U_j becomes very close to normal as n increase.

3. Comparisons

In this section, we compare the performance of the proposed floating charts with some recently proposed CUSUM and EWMA-type control charts for monitoring the process dispersion. The control charts selected for the comparison purpose include the EWMA $\ln - S^2$ by Castagliola,[12] EWMA $J-S^2$ by Castagliola et al.[14] and CUSUM $-S^2$ by Castagliola et al.[13] directly, while we have also compared the performance of our proposed charts with the Shewhart $R-$ chart, a CUSUM chart for process dispersion proposed by Page[3] and an EWMA chart proposed by Crowder and Hamilton,[8] indirectly.

3.1. Proposed versus EWMA **ln** $-S^2$ and EWMA $J-S^2$

Castagliola[12] proposed an EWMA chart for monitoring the process dispersion based on the same logarithmic transformation as in (2), named as EWMA $\ln - S^2$. Following him, Castagliola et al.[14] proposed another EWMA chart based on the same four parameter Johnson S_B transformation as in (3), named as EWMA $J-S^2$ for controlling the process standard deviation. The two parameters of these charts are the smoothing parameter λ and the control limit coefficient K. The *ARL* values of these two charts for the optimal choices of λ and K are given in Table VIII.

Comparing the performance of the proposed charts (having $q=0.3$) with these EWMA-type charts, we notice that both proposed charts have smaller ARL_1 values for a fixed $ARL_0=370$. Moreover, the proposed charts are showing more dominance for the smaller shifts as compared to the larger values of δ (cf. Tables IV and VII vs. Table VIII).

Castagliola[12] showed in his article that the EWMA $\ln - S^2$ control chart performs better than the Shewhart $R-$ chart for small shifts like $\delta \le 2$. He also proved the dominance of his proposed chart over the CUSUM chart proposed by Page[3] and the EWMA charts proposed by Crowder and Hamilton.[8] Therefore, we can state that the performance of our proposed charts is better than these charts also.

3.2. Proposed versus CUSUM $-S^2$

Castagliola et al.[13] proposed a CUSUM $-S^2$ chart based on the three-parameter logarithmic transformation as in (2). The sensitivity parameter is denoted by L and the control limit coefficient is represented by K. The *ARLs* of the CUSUM $-S^2$ chart with optimal parameter choices are given in Table IX.

The performance of this CUSUM $-S^2$ control chart is more or less similar to that of EWMA $\ln - S^2$ and EWMA $J-S^2$ charts. Comparing the performance of our proposed charts (having $q=0.3$) with the CUSUM $-S^2$, we may conclude that the proposed charts are performing better than the CUSUM $-S^2$ chart for almost all the values of δ (cf. Tables IV and VII vs. Table IX).

Table VIII. *ARL* values for the EWMA **ln** $-S^2$ and EWMA $J-S^2$ charts with $ARL_0 = 370$

δ	EWMA **ln** $-S^2$				EWMA $J-S^2$			
	$n=3$	$n=5$	$n=7$	$n=9$	$n=3$	$n=5$	$n=7$	$n=9$
0.5	10	5.6	4	3.1	9.4	5.2	3.7	2.9
0.6	13.9	8	5.7	4.5	13.6	7.7	5.5	4.3
0.7	21.3	12.6	9.1	7.1	21.2	12.4	8.9	7
0.8	40.8	23.1	17	13.5	40.8	23	16.8	13.4
0.9	130.3	68.9	48.5	38.2	126.5	68.3	48.4	38.1
0.95	289.7	184.8	137.6	110.3	274.3	179.9	135.8	109.4
1.05	173.2	142.3	115.1	96.8	195.9	148	118.1	98.8
1.1	91.9	59.8	45.2	36.9	97	61.6	46.1	37.5
1.2	36.5	22.8	17	13.8	38.8	23.5	17.4	14
1.3	20.3	12.4	9.3	7.5	21.3	13	9.6	7.7
1.4	13.4	8.1	6.1	4.9	13.8	8.4	6.3	5.1
1.5	9.8	5.8	4.4	3.6	9.8	6	4.5	3.7
2	3.9	2.3	1.8	1.5	3.8	2.3	1.8	1.5

Table IX. *ARL* values for the CUSUM $-S^2$ chart with $ARL_0 = 370$

δ	$n=3$	$n=5$	$n=7$	$n=9$
0.5	10.8	5.6	3.8	2.9
0.6	15.4	8.3	5.7	4.4
0.7	24.1	13.4	9.4	7.3
0.8	44	25.4	18.2	14.3
0.9	108.9	68.4	51.1	41.3
0.95	216.9	154.8	122.9	103.1
1.05	183.3	145.7	117.5	99.5
1.1	98.6	64.6	49.2	40.3
1.2	39.5	24.3	18	14.5
1.3	21.7	13.1	9.6	7.7
1.4	14.1	8.3	6.1	4.9
1.5	10.2	5.9	4.3	3.5
2	3.8	2.3	1.8	1.5

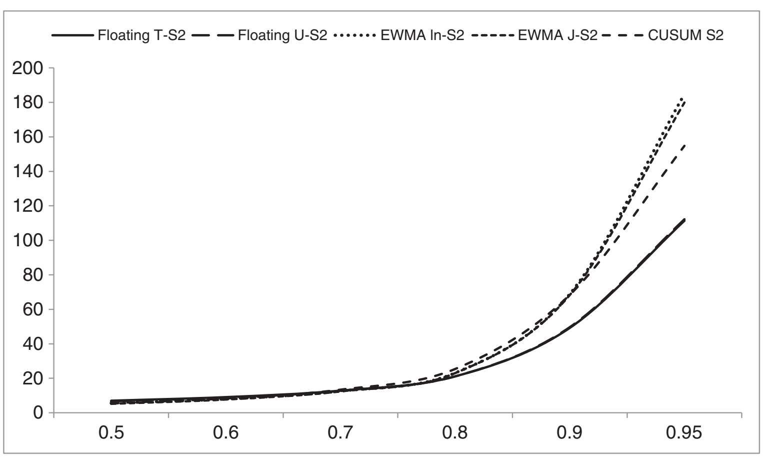

Figure 1. *ARL* curves for floating $T-S^2$, floating $U-S^2$, EWMA **ln** $-S^2$, EWMA $J-S^2$ and CUSUM S^2 charts for decrease in the process dispersion

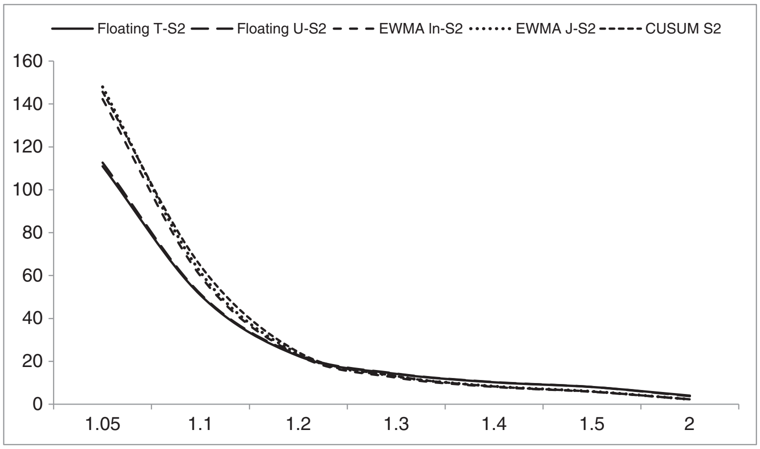

Figure 2. *ARL* curves for floating $T-S^2$, floating $U-S^2$, EWMA **ln** $-S^2$, EWMA $J-S^2$ and CUSUM S^2 charts for increase in the process dispersion

Apart from the tabular comparison, Figures 1 and 2 provide the *ARL* curves of the charts discussed in this Section for a decrease and an increase, respectively, in the process dispersion.

It is clear from Figures 1 – 2 that the *ARL* curves of both proposed charts are on the lower side of other curves. This shows that the proposed charts have smaller ARL_1 values for a fixed $ARL_0 = 370$. In addition, both proposed charts are showing almost same performance as their *ARL* curves are coinciding in both figures.

4. Illustrative example

Authors like Hawkins[4] and Thaga[20] suggested to provide an illustrative example in order to explain the implementation of the proposed chart. For the same purpose, we generate two datasets (namely dataset 1 and dataset 2) having 25 subgroups each of size

Table X. Calculation details of the proposed charts for dataset 1

Subgroup number	S_j^2	T_j	FT_j	LCL_{Tj}	UCL_{Tj}	U_j	FU_j	LCL_{Uj}	UCL_{Uj}
1	1.652	1.021	1.021	−5.942	5.956	1.011	1.011	−6.057	6.065
2	1.51	0.866	0.943	−3.409	3.424	0.864	0.937	−3.477	3.485
3	1.551	0.912	0.933	−2.463	2.478	0.908	0.928	−2.513	2.521
4	0.761	−0.172	0.657	−1.955	1.97	−0.136	0.662	−1.995	2.003
5	0.829	−0.057	0.514	−1.634	1.649	−0.022	0.525	−1.669	1.676
6	0.606	−0.458	0.352	−1.411	1.426	−0.425	0.367	−1.442	1.449
7	0.47	−0.742	0.196	−1.247	1.262	−0.723	0.211	−1.274	1.282
8	1.546	0.906	0.285	−1.12	1.135	0.902	0.297	−1.144	1.152
9	1.131	0.397	0.297	−1.018	1.033	0.418	0.311	−1.041	1.049
10	1.057	0.294	0.297	−0.935	0.95	0.318	0.311	−0.957	0.964
11	0.901	0.06	0.275	−0.866	0.881	0.093	0.292	−0.886	0.894
12	0.945	0.129	0.263	−0.807	0.822	0.159	0.281	−0.826	0.834
13	0.703	−0.275	0.222	−0.757	0.772	−0.239	0.241	−0.775	0.783
14	0.378	−0.955	0.138	−0.713	0.728	−0.957	0.155	−0.73	0.738
15	0.259	−1.263	0.044	−0.674	0.689	−1.315	0.057	−0.691	0.698
16	1.024	0.246	0.057	−0.64	0.655	0.273	0.071	−0.656	0.663
17	1.526	0.884	0.106	−0.609	0.624	0.881	0.118	−0.624	0.632
18	1.712	1.083	0.16	−0.582	0.597	1.07	0.171	−0.596	0.604
19	2.473	1.756	0.244	−0.557	0.572	1.722	0.253	−0.571	0.579
20	0.648	−0.377	0.213	−0.534	0.549	−0.342	0.223	−0.548	0.556
21	3.049	2.163	0.306	−0.513	0.528	2.135	0.314	−0.527	0.534
22	3.061	2.171	0.39	−0.494	0.509	2.143	0.397	−0.507	0.515
23	3.022	2.145	0.467	−0.477	0.492	2.116	0.472	−0.489	0.497
24	2.518	1.791	0.522[a]	−0.461	0.476	1.757	0.525[b]	−0.473	0.481
25	1.839	1.209	0.549[a]	−0.446	0.46	1.191	0.552[b]	−0.458	0.465

[a]indicates an out-of-control signal by floating $T-S^2$ chart
[b]indicates an out-of-control signal by floating $U-S^2$ chart

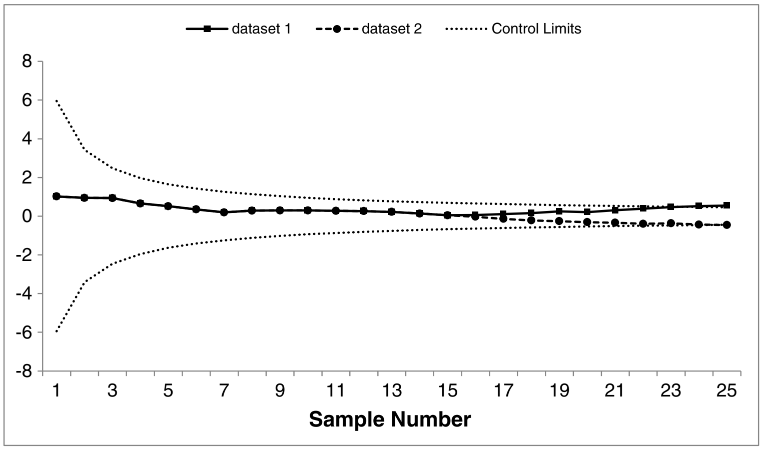

Figure 3. Chart output of floating $T-S^2$ chart for dataset 1 and 2

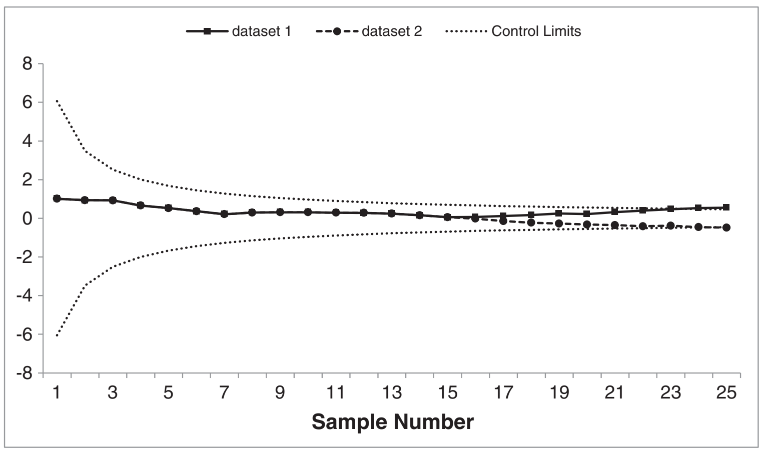

Figure 4. Chart output of floating $U-S^2$ chart for dataset 1 and 2

$n=5$, i.e. one for an increase and the other for a decrease in the process standard deviation. For dataset 1, the first 15 subgroups are generated from $N(0,1)$ showing an in-control standard deviation while the remaining 10 subgroups are generated from $N(0,1.3)$ referring to an out-of-control standard deviation with $\delta=1.3$. Similarly, for dataset 2 the first 15 subgroups are the same as for dataset 1, whereas the remaining 10 subgroups are taken from $N(0,0.7)$ showing an negative shift in the process dispersion with $\delta=0.7$. Both proposed charts are applied to the datasets with parameters; $\mu_T(n)=0.00748, \sigma_T(n)=0.967, A_T(n)=-0.8969, B_T(n)=2.3647, C_T(n)=0.5969, q=0.3$ and $K'_T=6.152$ for the proposed floating $T-S^2$ chart; $\mu_U(n)=0.0039, \sigma_U(n)=0.9852, A_U(n)=3.5402, B_U(n)=1.5727, C_U(n)=-0.2352, D_U(n)=11.312, q=0.3$ and $K'_U=6.152$ for the proposed floating $U-S^2$ chart. The calculations for both the proposed charts with dataset 1 are given in Table X. Figure 3 shows the chart output of the proposed floating $T-S^2$ chart for both datasets, while the chart output of floating $U-S^2$ chart is given in Figure 4.

It can be seen from Figures 3 – 4 that the proposed charts are effectively detecting both positive and negative shifts. This can also be confirmed from Table X, where both the proposed charts are signaling at subgroups # 24 and 25.

5. Summary and conclusions

In this article, we have proposed and studied two memory-type control charts, named as the floating $T-S^2$ control chart (based on a three-parameter logarithmic transformation) and the floating $U-S^2$ control chart (based on a four-parameter Johnson S_B transformation). The performance evaluation of the proposed charts is done by calculating the *ARL* and *SDRL* values using simulation procedures. These *ARLs* are compared with some EWMA- and CUSUM-type control charts for monitoring the process standard deviation. The comparisons show that the proposed charts are dominating the other charts under discussion in terms of *ARL* values. Moreover, an inter-proposed charts comparison shows that the floating $T-S^2$ chart is better for small shifts, whereas the floating $U-S^2$ chart is superior for large shifts in the process dispersion. At the end, an illustrative example is provided which shows the application of the proposed charts on simulated datasets.

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