Communications in Statistics—Theory and Methods, 43: 3485–3498, 2014

ISSN: 0361-0926 print / 1532-415X online
DOI: 10.1080/03610926.2012.700368

An EWMA-Type Control Chart for Monitoring the Process Mean Using Auxiliary Information

NASIR ABBAS,[1] MUHAMMAD RIAZ,[2]
AND RONALD J. M. M. DOES[3]

[1]Department of Basic Sciences, Riphah International University, Islamabad, Pakistan
[2]Department of Mathematics and Statistics, King Fahad University of Petroleum and Minerals, Dhahran, Saudi Arabia
[3]Department of Operations Management, IBIS UvA, University of Amsterdam, Amsterdam, The Netherlands

Statistical process control (SPC) is an important application of statistics in which the outputs of production processes are monitored. Control charts are an important tool of SPC. A very popular category is the Shewhart's $\bar{X}$-chart used to monitor the mean of a process characteristic. Two alternatives to the Shewhart's $\bar{X}$-chart are the cumulative sum and exponentially weighted moving average (EWMA) charts which are designed to detect moderate and small shifts in the process mean. Targeting on small and moderate shifts in the process mean, we propose an EWMA-type control chart which utilizes a single auxiliary variable. The regression estimation technique for the mean is used in defining the control structure of the proposed chart. It is shown that the proposed chart is performing better than its univariate and bivariate competitors which are also designed for detecting small shifts.

Keywords Auxiliary variable; Average run length (*ARL*); Control charts; Cumulative sum; Exponentially weighted moving average; M_XEWMA; Regression estimator.

Mathematics Subject Classification Primary 62P30; Secondary 65C05.

1. Introduction

Statistical process control (SPC) is broadly used in the manufacturing industry to attain the process stability and to improve the quality of the output of the process by reducing the amount of variation in it. The variation in the output is classified into common cause and special cause variations. The former is the essential part of any production process and in presence of only this variation, the process is said to be statistically in-control (cf. Montgomery, 2009). The latter is the type of variation which is targeted by SPC and most

Received March 21, 2011; Accepted June 1, 2012.
Address correspondence to Nasir Abbas, Assistant Professor, Department of Basic Sciences, Riphah International University, I-14 Islamabad, Pakistan; E-mail: nasirabbas55@yahoo.com

of the time there are assignable causes associated with this type of variation. A process working under both types of variation is said to be out-of-control and SPC techniques are available to detect the presence of special cause variation. Out of these techniques the control chart is the most important and most widely used.

Control charts are mainly classified into two categories (with respect to their design structure) named as memoryless (Shewhart-type) and memory control charts. Cumulative sum (CUSUM) control charts proposed by Page (1954) and exponentially weighted moving average (EWMA) control charts proposed by Roberts (1959) are the two most commonly used memory control charts. Their control structure is made so that (instead of ignoring the past information like Shewhart-type charts) they utilize the past information along with the current to give a better performance for small and moderate shifts. Details regarding the structure of the CUSUM charts and their averagerunlength(*ARL*) performance for different choices of parameters may be found in Hawkins and Olwell (1998) and references therein. Particulars of the EWMA control charts are given in detail in Sec. 2.

The concept of using auxiliary information is frequently used in the field of survey sampling and estimation techniques. Information accessible at the estimation stage other than that of sampled information is known as auxiliary information and regression estimation is one of the most efficient ways of using the auxiliary information. This auxiliary information is also used in control charting techniques in order to enhance their performance. Examples are the regression control chart proposed by Mandel (1969) and cause-selecting control charts proposed by Zhang (1985). The control structure of these control charts is based on regressing the study variable on the auxiliary variable. The residuals obtained from that regression are used for monitoring the process (see also Wade and Woodall, 1993).

Riaz (2008a) introduced the concept of using auxiliary information at the time of estimating the plotting statistic of a control chart. He proposed a control chart which uses a regression-type estimator as the plotting statistic to monitor the variability of the process and showed the dominance (in terms of power) of his proposed control chart over the well-known Shewhart-type control charts for the same purpose (i.e., R, S and S^2 charts). Riaz (2008b) proposed a regression-type estimator to monitor the location of the process. He not only showed the superiority of his proposal over the Shewhart's $\bar{X}$-chart but also over the regression charts (cf. Mandel, 1969) and the cause-selecting charts (cf. Zhang, 1985). In this article, we introduce an EWMA-type control chart, named as $\mathrm{M_X}$EWMA, based on a regression estimator using a single auxiliary variable for monitoring the location of the process. The performance of the proposed chart is evaluated in terms of *ARL* where *ARL* is defined as the expected number of samples until a shift has been detected. ARL_0 and ARL_1 are the in-control and out-of-control *ARLs*, respectively. For more details on *ARL* and other performance measures see Frisen (2003).

Section 2 contains the details of the basic structure of the classical EWMA. The design of the proposed $\mathrm{M_X}$EWMA and its performance evaluation is provided in Sec. 3. Comparison of the proposed chart with some recent CUSUM- and EWMA-type control charts is presented in Sec. 4. Also a comparison is made with a bivariate EWMA chart. To demonstrate the application of the proposed chart in real situations, an illustrative example is provided in Sec. 5 and finally the article is summarized in Sec. 6.

2. The Classical EWMA Control Chart

Let X represent a random variable from a normal distribution with mean μ_X and standard deviation σ_X. Then the EWMA statistic (cf. Roberts, 1959) for monitoring the process mean is defined as

$$Z_i = \gamma \bar{X}_i + (1-\gamma) Z_{i-1}, \tag{1}$$

where i is the sample number, γ is known as the smoothing constant and is chosen such that $0 < \gamma \leq 1$, and $\bar{X}_i$ is the average of the ith sample. The sizes of the samples are all equal to n. The quantity Z_{i-1} represents the past information and its initial value (i.e., Z_0) is taken equal to the target mean or the average of the preliminary samples. The parameter γ determines the rate at which past information comes into the calculation of the EWMA statistic. A large value of γ gives more weight to the current information and less weight to the past information, whereas a small value of γ gives more weight to the past information and less weight to the current information. $\gamma = 1$ implies that only the most recent observation is used for the calculation of the EWMA statistic and then the EWMA coincides with well-known Shewhart $\bar{X}$-chart. The in-control mean and the variance of the EWMA statistic (cf. Roberts, 1959) are

$$E(Z_i) = \mu_0, \quad V(Z_i) = \sigma_{\bar{X}}^2 \left(\frac{\gamma}{2-\gamma} (1 - (1-\gamma)^{2i}) \right), \tag{2}$$

where μ_0 represents the target mean of X and $\sigma_{\bar{X}}$ represents the standard deviation of $\bar{X}$, i.e., $\sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}}$. In case either or both μ_0 and $\sigma_{\bar{X}}$ are unknown, they may be estimated from the preliminary samples. Hence the lower control limit (LCL), the center line (CL) and the upper control limit (UCL) of the EWMA chart are given as

$$\left.\begin{array}{c} LCL_i = \mu_0 - L\sigma_{\bar{X}} \sqrt{\dfrac{\gamma}{2-\gamma} \left(1 - (1-\gamma)^{2i}\right)} \\ CL = \mu_0 \\ UCL_i = \mu_0 + L\sigma_{\bar{X}} \sqrt{\dfrac{\gamma}{2-\gamma} \left(1 - (1-\gamma)^{2i}\right)} \end{array}\right\}, \tag{3}$$

where L determines the width of the control limits for the EWMA chart (i.e., it is used to control the in-control *ARL* of the EWMA chart) and its value is chosen according to the tables given by Steiner (1999). From (3) it can be seen that the control limits are dependent of the number of samples used. The relation implies that the control limits are narrow for the initial samples and as time goes on, these limits converge toward constant levels given as

$$LCL = \mu_0 - L\sigma_{\bar{X}} \sqrt{\frac{\gamma}{2-\gamma}}, \quad CL = \mu_0, \quad UCL = \mu_0 + L\sigma_{\bar{X}} \sqrt{\frac{\gamma}{2-\gamma}}. \tag{4}$$

The control limits given in (3) are known as time varying limits while the limits given in (4) are called constant limits. The time varying limits use the exact width of the control limits for the initial samples and therefore an EWMA control chart using time varying limits will be more powerful than that of using the constant limits to detect the initial out-of-control points (cf. Montgomery, 2009). The *ARL* values for the classical EWMA with time varying limits in (3) (cf. Steiner, 1999) are given in Table 1. Here δ represents the amount of shift (defined as $\delta = \sqrt{n}|\mu_1 - \mu_0|/\sigma_X$ where μ_0 and μ_1 are the in-control and the out-of-control mean, respectively) and ARL_0 is fixed at 500.

Many modifications of the CUSUM and EWMA control charts are suggested in order to get smaller ARL_1 values for a fixed in-control ARL_0. These modifications include the first

Table 1
ARL values for the classical EWMA chart with time varying limits and $ARL_0 = 500$

δ	$\gamma = 0.03$ $L = 2.483$	$\gamma = 0.05$ $L = 2.639$	$\gamma = 0.1$ $L = 2.824$	$\gamma = 0.25$ $L = 3.001$	$\gamma = 0.5$ $L = 3.072$	$\gamma = 0.75$ $L = 3.088$
0	500	500	500	500	500	500
0.25	66.54	77.75	103.3	169.5	254.7	321.3
0.5	21.23	23.71	28.81	47.38	88.47	140.3
0.75	10.75	11.87	13.61	19.32	35.59	62.46
1	6.64	7.31	8.21	10.41	17.18	30.53
1.5	3.45	3.77	4.17	4.78	6.27	9.81
2	2.24	2.43	2.66	2.94	3.39	4.46
2.5	1.66	1.77	1.92	2.09	2.26	2.62
3	1.34	1.41	1.51	1.62	1.70	1.82
4	1.07	1.09	1.12	1.16	1.18	1.20
5	1.01	1.01	1.01	1.02	1.03	1.03

initial response (FIR) CUSUM and FIR EWMA presented by Lucas and Crosier (1982) and Rhoads et al. (1996), respectively. The idea of the FIR feature is to assign a head start to the initial value of the statistic rather than setting it equal to zero. Recently, Riaz et al. (2011) and Abbas et al. (2011) proposed the use of the runs rules schemes with the CUSUM and EWMA charts, respectively. They show that the CUSUM and EWMA charts with the extra runs rules schemes were superior to the FIR CUSUM and FIR EWMA charts, respectively. We propose, in the next section, an M_XEWMA control chart which uses the information of a single auxiliary variable on the regression estimator's pattern to monitor the location of a process and without loss of generality we have considered the case of individual observations.

3. The Proposed M_XEWMA Control Chart

Let an auxiliary variable W_i be correlated with the variable of interest X_i and let us denote the correlation between these two variables by ρ_{XW}. The observations of X and W are obtained in the paired form for each sample and the population mean and variance of W (i.e., μ_W and σ_W^2, respectively) are assumed to be known. Also we assume bivariate normality of X and W, i.e., $(X, W) \sim N_2(\mu_X, \mu_W, \sigma_X^2, \sigma_W^2, \rho_{XW})$ where N_2 represents the bivariate normal distribution. The regression estimate of the population mean μ_X (cf. Cochran, 1977) is given as

$$M_X = \bar{X} + b_{XW}(\mu_W - \bar{W}), \tag{5}$$

where b_{XW} is the change in X due to one unit change in W and is $b_{XW} = \rho_{XW}(\frac{\sigma_X}{\sigma_W})$. The mean and variance of the statistic M_X are given as

$$E(M_X) = \mu_X, \quad V(M_X) = \sigma_M^2 = \frac{\sigma_X^2}{n}\left(1 - \rho_{XW}^2\right) = \frac{\sigma_X^2 - b_{XW}^2\sigma_W^2}{n}. \tag{6}$$

Equation (6) implies that M_X is also an unbiased estimator of μ_X (like $\bar{X}$) and $V(M_X) > V(\bar{X})$ as long as $\rho_{XW}^2 > 0$. Based on the regression estimator in (5), the plotting statistic for the proposed M_XEWMA chart is defined as

$$Y_i = \gamma_M M_{X_i} + (1 - \gamma_M) Y_{i-1}, \tag{7}$$

where γ_M is the smoothing constant for the proposed statistic and M_{X_i} is the value of statistic M_X for the ith sample. Y_{i-1} represents the past information (like Z_{i-1} in (1)) and its initial value (i.e., Y_0) is also taken equal to the target mean μ_0. Now based on (6) the time varying control limits for the proposed chart are

$$\left.\begin{array}{c} LCL_i = \mu_0 - L_M \sigma_M \sqrt{\dfrac{\gamma_M}{2-\gamma_M}\left(1-(1-\gamma_M)^{2i}\right)} \\ CL = \mu_0 \\ UCL_i = \mu_0 + L_M \sigma_M \sqrt{\dfrac{\gamma_M}{2-\gamma_M}\left(1-(1-\gamma_M)^{2i}\right)} \end{array}\right\}, \tag{8}$$

where L_M determines the width of the control limits for the proposed M_XEWMA chart. Following Steiner (1999) we have also done a complete performance evaluation of the proposed chart in terms of *ARL* which are calculated through a Monte Carlo simulation procedure by running 50,000 replications. The *ARL* values for the proposed M_XEWMA chart with time varying limits are given in Tables 2–6 for some selective choices of ρ_{XW} in which δ represents the amount of shift in the study variable, where μ_W remains constant.

The ARL_0 in Tables 2–6 is fixed at 500 which will enable us to make comparison of the proposed control chart with some other charts/schemes. Tables 2–6 refer to a situation where the information about the population correlation coefficient is assumed to be known, because the information about the population correlation coefficient ρ_{XW} is known in many practical situations (cf. Garcia and Cebrian, 1996).

Riaz et al. (2011) and Abbas et al. (2011) suggested reporting the relative standard errors of the results so we replicated some of the results in Tables 2–6 repeatedly and found that the relative standard errors of the results are about 0.8%.

Table 2
ARL values for the proposed M_XEWMA chart with time varying limits, $\rho_{XW} = 0.05$ and $ARL_0 = 500$

δ	$\gamma_M = 0.03$ $L_M = 2.483$	$\gamma_M = 0.05$ $L_M = 2.639$	$\gamma_M = 0.1$ $L_M = 2.824$	$\gamma_M = 0.25$ $L_M = 3.001$	$\gamma_M = 0.5$ $L_M = 3.072$	$\gamma_M = 0.75$ $L_M = 3.088$
0	500.3011	500.8313	499.7023	499.6045	500.9026	499.9704
0.25	66.3103	77.5273	103.1312	168.8081	254.324	321.6227
0.5	21.2101	23.6211	28.7434	47.2173	88.1445	139.8713
0.75	10.7125	11.8433	13.592	19.2345	35.401	62.2502
1	6.632	7.2905	8.2072	10.3617	17.1148	30.3995
1.5	3.4443	3.7573	4.1644	4.7647	6.2628	9.7633
2	2.244	2.4205	2.6563	2.9313	3.3701	4.4521
2.5	1.6555	1.7719	1.9147	2.0831	2.2565	2.6079
3	1.3401	1.4104	1.5112	1.6109	1.6904	1.8151
4	1.0642	1.0882	1.1224	1.1609	1.184	1.1946
5	1.0049	1.0085	1.0131	1.0221	1.0265	1.0281

Table 3
ARL values for the proposed M_XEWMA chart with time varying limits, $\rho_{XW} = 0.25$ and $ARL_0 = 500$

δ	$\gamma_M = 0.03$ $L_M = 2.483$	$\gamma_M = 0.05$ $L_M = 2.639$	$\gamma_M = 0.1$ $L_M = 2.824$	$\gamma_M = 0.25$ $L_M = 3.001$	$\gamma_M = 0.5$ $L_M = 3.072$	$\gamma_M = 0.75$ $L_M = 3.088$
0	500.0592	500.5358	499.1371	499.8049	499.951	500.5678
0.25	63.1554	73.8038	97.5858	161.9697	245.9632	313.2722
0.5	20.1268	22.4316	27.1393	44.1797	83.0098	132.8202
0.75	10.1982	11.2488	12.8277	18.0008	32.9028	57.9059
1	6.3049	6.9254	7.7548	9.7748	15.8281	28.0123
1.5	3.2871	3.5816	3.9532	4.5005	5.832	8.9444
2	2.1384	2.3119	2.5261	2.7962	3.1754	4.0979
2.5	1.5945	1.6997	1.8406	1.9877	2.1497	2.4354
3	1.2952	1.363	1.4519	1.5523	1.6178	1.7158
4	1.0495	1.0687	1.0955	1.132	1.1476	1.1565
5	1.0044	1.0067	1.0108	1.0149	1.0178	1.0183

The main findings about our proposed M_XEWMA control chart for monitoring the location of a process are given as (cf. Tables 2–6):

i. the use of auxiliary information in the form of a regression estimator boosts the performance of EWMA control chart, especially for the larger values of ρ_{XW};
ii. for the fixed values of ρ_{XW} and δ, the performance of the proposed chart with time varying limits is better for smaller values of γ_M;
iii. for the fixed values of γ_M, L_M and δ, the performance of the proposed chart is better for the larger values of ρ_{XW};

Table 4
ARL values for the proposed M_XEWMA chart with time varying limits, $\rho_{XW} = 0.5$ and $ARL_0 = 500$

δ	$\gamma_M = 0.03$ $L_M = 2.483$	$\gamma_M = 0.05$ $L_M = 2.639$	$\gamma_M = 0.1$ $L_M = 2.824$	$\gamma_M = 0.25$ $L_M = 3.001$	$\gamma_M = 0.5$ $L_M = 3.072$	$\gamma_M = 0.75$ $L_M = 3.088$
0	500.7792	499.5635	499.8114	499.692	500.7859	500.9686
0.25	52.6523	61.1011	80.6591	135.7683	216.0974	285.4555
0.5	16.6817	18.5391	22.044	34.5785	65.3989	108.1412
0.75	8.4242	9.3261	10.5432	14.1099	24.7989	44.3004
1	5.2477	5.7592	6.4306	7.8001	11.9099	20.6193
1.5	2.7663	3.0129	3.3183	3.7182	4.5573	6.5619
2	1.8392	1.9673	2.1455	2.3486	2.5852	3.136
2.5	1.3979	1.48	1.5834	1.7044	1.7964	1.9476
3	1.17	1.2168	1.2786	1.3501	1.3947	1.4325
4	1.0155	1.0253	1.038	1.0513	1.0606	1.064
5	1.0007	1.0012	1.0019	1.002	1.0028	1.0056

Table 5
ARL values for the proposed M_XEWMA chart with time varying limits, $\rho_{XW} = 0.75$ and $ARL_0 = 500$

δ	$\gamma_M = 0.03$ $L_M = 2.483$	$\gamma_M = 0.05$ $L_M = 2.639$	$\gamma_M = 0.1$ $L_M = 2.824$	$\gamma_M = 0.25$ $L_M = 3.001$	$\gamma_M = 0.5$ $L_M = 3.072$	$\gamma_M = 0.75$ $L_M = 3.088$
0	500.0659	499.5868	500.7509	500.4051	499.2859	499.7069
0.25	33.8532	38.4587	48.8401	84.1265	146.8905	211.6496
0.5	10.6126	11.7078	13.4213	18.9673	34.9019	61.2737
0.75	5.398	5.9343	6.6291	8.0816	12.4585	21.7022
1	3.4068	3.7177	4.1045	4.7175	6.1522	9.5862
1.5	1.8835	2.0268	2.2095	2.4126	2.6787	3.2731
2	1.3299	1.4043	1.4969	1.5949	1.6784	1.7904
2.5	1.1006	1.1309	1.1772	1.2269	1.2577	1.2721
3	1.0199	1.0296	1.0429	1.0608	1.0733	1.0742
4	1.0001	1.0004	1.0006	1.0011	1.0014	1.0017
5	1	1	1	1	1	1

iv. for all the choices of γ_M, L_M and ρ_{XW} the proposed chart is *ARL* unbiased, i.e., ARL_1 never exceeds ARL_0 for any value of δ;

v. for smaller values of ρ_{XW}, the *ARL* for the proposed scheme decreases gradually with an increase in the value of δ but for the larger values of ρ_{XW}, the *ARL* for the proposed scheme decreases rapidly with an increase in the value of δ.

4. Comparisons

In this section, we provide a broad comparison of our proposed M_XEWMA chart with the classical CUSUM, the classical EWMA, some of their recent modifications in which *ARL*

Table 6
ARL values for the proposed M_XEWMA chart with time varying limits, $\rho_{XW} = 0.95$ and $ARL_0 = 500$

δ	$\gamma_M = 0.03$ $L_M = 2.483$	$\gamma_M = 0.05$ $L_M = 2.639$	$\gamma_M = 0.1$ $L_M = 2.824$	$\gamma_M = 0.25$ $L_M = 3.001$	$\gamma_M = 0.5$ $L_M = 3.072$	$\gamma_M = 0.75$ $L_M = 3.088$
0	500.1483	499.3424	500.5895	500.2327	500.0067	500.7743
0.25	9.6039	10.6139	12.1289	16.7024	30.2287	53.736
0.5	3.1274	3.4017	3.7577	4.2576	5.389	8.1474
0.75	1.75	1.8739	2.0315	2.2142	2.4181	2.8603
1	1.2547	1.3136	1.3951	1.4846	1.5468	1.622
1.5	1.0102	1.0162	1.0216	1.034	1.0418	1.0432
2	1	1.0003	1.003	1.0005	1.0006	1.0013
2.5	1	1	1	1	1	1
3	1	1	1	1	1	1
4	1	1	1	1	1	1
5	1	1	1	1	1	1

is the performance measure used for comparison purposes, and the bivariate EWMA chart introduced by Lowry et al. (1992). Below, we present the one by one comparison of the proposed scheme with its counterparts:

4.1. M_XEWMA vs. Classical EWMA

The classical EWMA chart proposed by Roberts (1959) is discussed comprehensively in Sec. 2. *ARLs* for the classical EWMA chart with time varying are given in Table 1. Comparing the M_XEWMA chart (with $\gamma_M = 0.25$) with the classical EWMA chart we can see that for all the values of ρ_{XW} the *ARL* performance of the M_XEWMA chart is better than the classical EWMA for a fixed value of δ (cf. Table 1 vs. Tables 2–6). Moreover, an important point here is that the classical EWMA is a special case of the M_XEWMA chart, i.e., applying M_XEWMA chart to a process where $\rho_{XW} = 0$ is equivalent to applying the classical EWMA. From Table 2, we see that for $\rho_{XW} = 0.25$ the results almost coincides with the results of Table 1 as was to be expected.

It is to be noted here that the results of proposed M_XEWMA chart with time varying limits are on the same pattern as compared to the classical EWMA with time varying limits while the proposed chart with asymptotic limits (computational results are not provided here) is mainly following the pattern of classical EWMA with asymptotic limits.

4.2. M_XEWMA vs. Classical CUSUM

The classical CUSUM proposed by Page (1954) is discussed briefly in Sec. 1. A comprehensive study on the CUSUM charts is given by Hawkins and Olwell (1998). The *ARLs* for the CUSUM chart are given in Table 7 with ARL_0 fixed at 500.

Comparing the M_XEWMA chart with the classical CUSUM chart we observe that the newly proposed chart is outperforming the classical CUSUM even for the smaller values of ρ_{XW} (cf. Table 7 vs. Tables 2–6).

4.3. M_XEWMA vs. Runs Rules Based CUSUM and EWMA

Riaz et al. (2011) introduced the concept of applying runs rules with the CUSUM charts followed by Abbas et al. (2011) who applied two runs rules schemes with the EWMA charts. The *ARLs* for the runs rules based CUSUM and EWMA are given in Tables 8 and 9, respectively.

Table 7
ARL values for the classical CUSUM scheme

		δ										
k	h	0	0.25	0.5	0.75	1	1.5	2	2.5	3	4	5
0.25	8.585	500	94.8	31.08	17.54	12.17	7.58	5.55	4.41	3.69	2.84	2.26
0.5	5.071	500	145.5	38.87	17.32	10.52	5.82	4.06	3.15	2.6	2.03	1.72
0.75	3.539	500	200.7	57.07	22.13	11.6	5.45	3.55	2.67	2.18	1.63	1.24
1	2.665	500	249.5	81.44	30.9	14.67	5.75	3.41	2.45	1.94	1.38	1.09

Table 8
ARL_1 values for the runs rules based CUSUM at $ARL_0 = 500$

	Limits		δ					
Scheme	WL	AL	0.25	0.5	0.75	1	1.5	2
I	4.8	5.12	141.1114	38.5986	17.3916	10.5176	5.9052	4.0574
	4.7	5.2	150.3718	38.5942	17.5288	10.5994	5.8984	4.14
	4.6	5.39	145.1886	38.1954	17.468	10.5584	6.0066	4.2374
	4.49	∞	146.564	38.4918	17.7254	10.8566	6.3326	4.6894
II	4.8	5.11	139.7048	38.8562	17.4586	10.5056	5.8222	4.0776
	4.7	5.19	142.1588	37.9752	17.2674	10.5826	5.8716	4.1036
	4.6	5.5	145.7868	38.3342	17.3938	10.734	6.0526	4.2726
	4.54	∞	149.0352	39.9042	17.5682	10.9658	6.4506	4.873

WL and AL in Table 8 represent the warning and action limits, respectively (for more details cf. Riaz et al., 2010). Comparing the M_XEWMA chart with runs rules based CUSUM and EWMA, we can see that the proposed chart is uniformly surpassing both the CUSUM schemes I and II and EWMA scheme I. The EWMA scheme II is performing better than the proposed chart as long as $\rho_{XW} < 0.5$ but once we have $\rho_{XW} \geq 0.5$, the proposed chart outperforms the EWMA scheme II as well (cf. Table 8 and 9 vs. Tables 2–6).

4.4. M_XEWMA vs. MEWMA

Lowry et al. (1992) introduced a multivariate extension of the EWMA chart named as MEWMA chart. For the bivariate case, the MEWMA statistic is

$$\begin{pmatrix} Z_i \\ Y_i \end{pmatrix} = r \begin{pmatrix} X_i \\ W_i \end{pmatrix} + (1-r) \begin{pmatrix} Z_{i-1} \\ Y_{i-1} \end{pmatrix},$$

where r is the smoothing constant. The chart gives an out-of-control signal if $T_i^2 = C_1 \begin{pmatrix} Z_i & Y_i \end{pmatrix} \begin{pmatrix} \sigma_W^2 & -\sigma_{XW} \\ -\sigma_{WX} & \sigma_X^2 \end{pmatrix} \begin{pmatrix} Z_i \\ Y_i \end{pmatrix} > h_4$. Here $C_1 = \frac{2-r}{r(1-(1-r)^{2i})\sigma_X^2 \sigma_W^2 (1-\rho_{XW}^2)}$ and h_4 is the *UCL*.

Table 9
ARL values for the runs rules based EWMA at $ARL_0 = 500$

	Scheme I			Scheme II		
δ	$\lambda = 0.1$ $L_s = 2.556$	$\lambda = 0.25$ $L_s = 2.554$	$\lambda = 0.5$ $L_s = 2.36$	$\lambda = 0.1$ $L_s = 2.3$	$\lambda = 0.25$ $L_s = 2.345$	$\lambda = 0.5$ $L_s = 2.002$
0	501.7558	505.5284	501.2598	502.883	499.6153	505.3564
0.25	103.3109	169.1349	235.1138	66.6864	97.0108	133.7117
0.5	29.5748	47.0105	78.0771	21.4251	31.2023	46.3541
0.75	14.3216	19.2776	30.8742	11.7427	14.4295	20.6223
1	8.9561	10.5964	15.1992	7.5539	8.6761	11.0991
1.5	4.9197	5.2578	6.1014	4.4676	4.7066	5.1336
2	3.4498	3.5527	3.6815	3.4534	3.549	3.6276

Table 10
ARL Values for MEWMA charts with $r = 0.1$ and $h_4 = 10.833$ at $ARL_0 = 500$

δ	$\rho_{XW} = 0.05$	$\rho_{XW} = 0.25$	$\rho_{XW} = 0.5$	$\rho_{XW} = 0.75$	$\rho_{XW} = 0.95$
0	500.57	500.6147	500.9223	499.2386	500.4125
0.5	36.81699	34.50276	27.63295	16.425	4.41316
1	9.8781	9.31899	7.66392	4.85723	1.55707
1.5	4.91344	4.65957	3.88276	2.54956	1.05029
2	3.08706	2.94745	2.47956	1.6849	1.00066
2.5	2.19454	2.10018	1.79616	1.27854	1.00001
3	1.6936	1.6269	1.41069	1.08457	1

Lowry et al. (1992) assume that the mean of both variables are equal (i.e., $\mu_X = \mu_W = \mu$). According to this assumption the *ARL* values of MEWMA only depend upon the shift parameter $\lambda = \sqrt{\frac{2}{1+\rho_{XW}}}\mu$. We have developed a code in R language to evaluate the *ARL* values of this bivariate EWMA chart through Monte Carlo simulation. To validate our simulation code, we have replicated the results of Table 1 (by running 100,000 replications) of Lowry et al. (1992) article and found the same results.

Now using our simulation code, we find the *ARL* values of bivariate EWMA with the mean of W fixed and the shift in the mean of X as we have defined in Secs. 2 and 3. These *ARL*s are given in Table 10, where $r = 0.1$ and $h_4 = 10.833$ with $ARL_0 = 500$.

Comparing the performance of the bivariate EWMA with the proposed chart, it can be noticed that the proposed chart has smaller ARL_1 values as compared to the bivariate EWMA chart for all corresponding values of ρ_{XW} (cf. Table 10 vs. Tables 2–6).

5. Illustrative Example

It is recommended by the authors like Steiner (1999) and Lucas and Saccucci (1990) to provide an illustrative example of the proposed chart to show the application of the proposed chart in real situation. For this purpose, we generate a dataset containing 20 observations from $N_2(\mu_X + \delta\sigma_X, \mu_W, \sigma_X^2, \sigma_W^2, \rho_{XW})$. We use $\delta = 0.5$ (referring to an out-of-control situation with a 0.5 sigma shift in the study variable X), and μ_X and μ_W are taken equal to zero, whereas σ_X^2 and σ_W^2 are taken equal to unity for convenience. The correlation between the study and auxiliary variables is 0.5 (i.e., $\rho_{XW} = 0.5$). In the comparison we use the classical CUSUM, the classical EWMA and the M_XEWMA charts. The parameters for the classical CUSUM are selected as $k = 0.5$ and $h = 5.071$ (cf. Table 7); for the classical EWMA with time varying limits we have taken $\gamma = 0.1$ and $L = 2.824$ (cf. Table 1) and for the M_XEWMA with time varying limits we have used $\gamma_M = 0.1$ and $L_M = 2.824$ in order to obtain for all three charts an $ARL_0 = 500$. Table 11 contains the calculations done for the M_XEWMA chart, whereas the graphical display for the three control charts is given in Figs. 1–3. The plotting statistic for the classical CUSUM are C_i^+ and C_i^-; for the classical EWMA is Z_i and for the M_XEWMA is Y_i.

We can clearly see from Fig. 1 that the proposed M_XEWMA chart is giving out-of-control signals at samples # 18, 19 and 20. In total, we have three out-of-control signals

Table 11
Example of the proposed M_XEWMA control chart with time varying limits for dataset with known $\mu_0 = 0$, $\sigma_X = 1$, $\mu_W = 0$, $\sigma_W = 1$, $\rho_{XW} = 0.5$ and parameters of chart $\gamma_M = 0.1$ and $L_M = 2.824$ at $ARL_0 = 500$

Sample No	X_i	W_i	M_X	Y_i	UCL_i	LCL_i
1	0.39	−0.865	0.823	0.082	0.245	0.39
2	−0.242	−1.686	0.601	0.134	0.329	−0.242
3	−0.919	−1.046	−0.396	0.081	0.384	−0.919
4	−1.22	−1.366	−0.537	0.019	0.423	−1.22
5	2.01	0.574	1.722	0.19	0.453	2.01
6	1.395	1.61	0.59	0.23	0.475	1.395
7	1.66	1.542	0.889	0.296	0.493	1.66
8	−0.514	0.816	−0.922	0.174	0.506	−0.514
9	−0.213	−0.907	0.24	0.18	0.517	−0.213
10	−0.588	−1.923	0.374	0.2	0.526	−0.588
11	0.074	0.132	0.008	0.181	0.533	0.074
12	1.673	1.64	0.853	0.248	0.538	1.673
13	1.765	0.575	1.478	0.371	0.543	1.765
14	0.061	−0.008	0.065	0.34	0.546	0.061
15	1.537	−1.084	2.079	0.514	0.549	1.537
16	−0.519	−0.52	−0.259	0.437	0.551	−0.519
17	1.198	−0.246	1.32	0.525	0.553	1.198
18	1.853	0.028	1.839	0.656*	0.555	1.853
19	0.733	1.715	−0.125	0.578*	0.556	0.733
20	0.108	−0.6	0.408	0.561*	0.557	0.108

*indicates proposed chart giving out-of-control signal

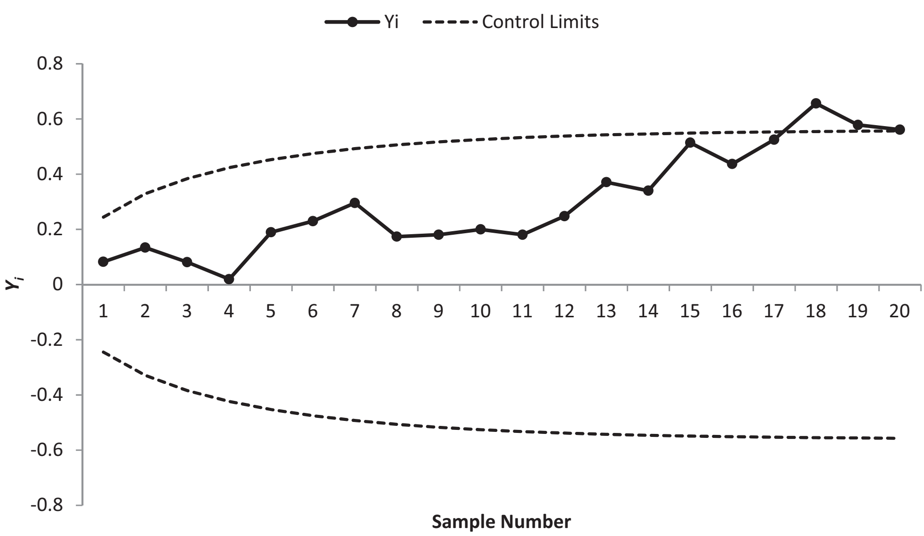

Figure 1. The proposed M_XEWMA control chart with time varing limits, known $\mu_0 = 0$, $\sigma_X = 1$, $\mu_W = 0$, $\sigma_W = 1$, $\rho_{XW} = 0.5$ and parameters of chart $\gamma_M = 0.1$ and $L_M = 2.824$ at $ARL_0 = 500$.

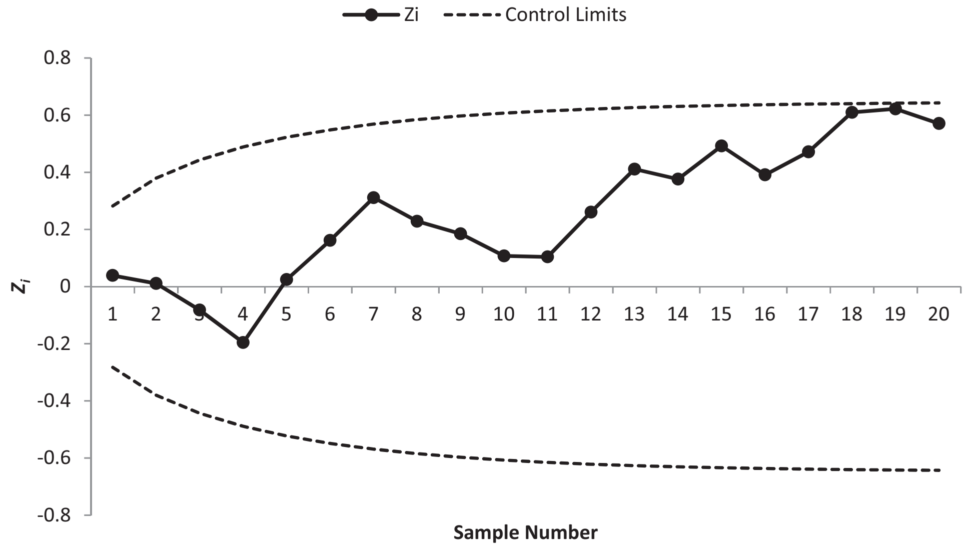

Figure 2. The classical EWMA control chart with time varying limits, known $\mu_0 = 0$, $\sigma_X = 1$, and parameters of chart $\gamma = 0.1$ and $L = 2.824$ at $ARL_0 = 500$.

which can also be confirmed through Table 11. Both the classical EWMA and the classical CUSUM failed to detect any shift in the process mean (cf. Figs. 2 and 3, respectively). This shows the superiority of the proposed chart over the classical EWMA and the classical CUSUM in the form of signaling earlier.

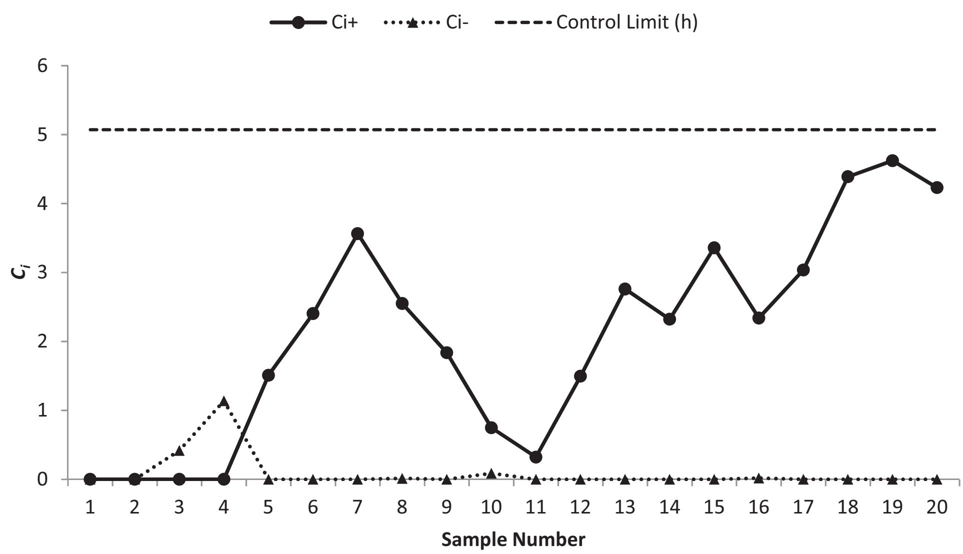

Figure 3. The classical CUSUM control chart with known $\mu_0 = 0$, $\sigma_X = 1$, and parameters of chart $k = 0.5$ and $h = 5.071$ at $ARL_0 = 500$.

6. Summary and Conclusions

Control charts are commonly used to monitor quality characteristics from manufacturing processes. Shewhart-type control charts are good at detecting the larger shifts in the process while CUSUM- and EWMA-type control charts are specialized to detect small and moderate shifts in the process. After the introduction of CUSUM and EWMA charts, many authors have proposed modifications of these two control charts in order to further enhance their performance. We have proposed, in this article, a new EWMA-type control chart, called the M_XEWMA control chart, which is based on a regression estimator using the information of a single auxiliary variable. Note that the proposed M_XEWMA chart is equal to the classical EWMA chart, when the correlation between the study and auxiliary variables is equal to 0. The performance of the proposed chart is evaluated in terms of *ARL* for different values of the correlation between the study variable and the auxiliary variable. A comparison of the proposed chart with the classical CUSUM, the classical EWMA and some of their univariate and bivariate modifications is also made. The comparisons showed that the proposed chart is good at detecting small to moderate shifts in the process location, while its ability to detect large shifts is not bad either.

Acknowledgment

The author Muhammad Riaz is indebted to King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia, for providing excellent research facilities.

References

Abbas, N., Riaz, M., Does, R. J. M. M. (2011). Enhancing the performance of EWMA charts. *Qual. Reliab. Eng. Int.*, 27(6):821–833.

Cochran, W. G. (1977). *Sampling Techniques* (3rd ed.). New York, NY: John Wiley & Sons.

Frisen, M. (2003). Statistical surveillance. Optimality and methods. *Int. Stat. Rev.*, 71(2):403–434.

Garcia, M. R., Cebrian, A. A. (1996). Repeated substitution method: The ratio estimator for the population variance. *Metrika*, 43:101–105.

Hawkins, D. M., Olwell, D. H. (1998). *Cumulative Sum Charts and Charting Improvement*. New York, NY: Springer.

Lowry, C. A., Woodall, W. H., Champ, C. W., Rigdon, S. E. (1992). A multivariate exponentially weighted moving average control chart. *Technometrics*, 34(1):46–53.

Lucas, J. M., Crosier, R. B. (1982). Fast initial response for CUSUM quality-control scheme: give your CUSUM a headstart. *Technometrics*, 24:199–205.

Lucas J. M., Saccucci, M. S. (1990). Exponentially weighted moving average control schemes: Properties and enhancements. *Technometrics*, 32:1–12.

Mandel, B. J. (1969). The regression control chart. *J. Qual. Technol.*, 1:1–9.

Montgomery, D. C. (2009). *Introduction to Statistical Quality Control* (6th ed.). New York, NY: John Wiley & Sons.

Page, E. S. (1954). Continuous inspection schemes. *Biometrika*, 41:100–115.

Rhoads, T. R., Montgomery, D. C., Mastrangelo, C. M. (1996). Fast initial response scheme for exponentially weighted moving average control chart. *Qual. Eng.*, 9(2):317–327.

Riaz, M. (2008a). Monitoring process variability using auxiliary information. *Comput. Stat.*, 23(2):253–276.

Riaz, M. (2008b). Monitoring process mean level using auxiliary information. *Statistica Neerlandica*, 62(4):458–481.

Riaz, M., Abbas, N., Does, R. J. M. M. (2011). Improving the performance of CUSUM charts. *Qual. Reliab. Eng. Int.*, 27(4):415–424.

Roberts, S. W. (1959). Control chart tests based on geometric moving averages. *Technometrics*, 1:239–250.

Steiner, S. H. (1999). EWMA control charts with time-varying control limits and fast initial response. *J. Qual. Technol.*, 31(1):75–86.

Wade, M. R., Woodall, W. H. (1993). A review and analysis of cause-selecting control charts. *J. Qual. Technol.*, 25(3):161–169.

Zhang, G. X. (1985). Cause-selecting control charts-a new type of quality control charts. *QR J.*, 12:221–225.