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Mixed Exponentially Weighted Moving Average–Cumulative Sum Charts for Process Monitoring

Nasir Abbas,[a*†] Muhammad Riaz[a,b] and Ronald J. M. M. Does[c]

The control chart is a very popular tool of statistical process control. It is used to determine the existence of special cause variation to remove it so that the process may be brought in statistical control. Shewhart-type control charts are sensitive for large disturbances in the process, whereas cumulative sum (CUSUM)–type and exponentially weighted moving average (EWMA)–type control charts are intended to spot small and moderate disturbances. In this article, we proposed a mixed EWMA–CUSUM control chart for detecting a shift in the process mean and evaluated its average run lengths. Comparisons of the proposed control chart were made with some representative control charts including the classical CUSUM, classical EWMA, fast initial response CUSUM, fast initial response EWMA, adaptive CUSUM with EWMA-based shift estimator, weighted CUSUM and runs rules–based CUSUM and EWMA. The comparisons revealed that mixing the two charts makes the proposed scheme even more sensitive to the small shifts in the process mean than the other schemes designed for detecting small shifts. Copyright © 2012 John Wiley & Sons, Ltd.

Keywords: average run length (ARL); control charts; cumulative sum; exponentially weighted moving average; statistical process control

1. Introduction

Variations in a process are classified into two distinct parts called common and special cause variations. In the presence of common cause variation only, the process is said to be statistically in control, but once the process includes both common and special cause variations, it is deemed out of control. Statistical process control (SPC) is the application of the statistical tools to distinguish between common and special cause variations (cf. Montgomery[1]). The most important of those statistical tools is control chart. Cumulative sum (CUSUM) charts by Page[2] and exponentially weighted moving average (EWMA) charts by Roberts[3] are the two most commonly used types of control charts for detecting the smaller and moderate shifts in the process, whereas Shewhart-type charts are good at detecting larger shifts.

An effective measure used for comparing the performance of the control charts is the average run length (ARL). If we define a random variable R_L equal to the number of samples until the first out of control signal occurs, then the probability distribution of this random variable R_L is known as the run length distribution. The average of this distribution is the ARL. The in-control ARL of a control chart is denoted by ARL_0, whereas out-of-control ARL is denoted by ARL_1.

Lucas[4] proposed the use of a combined Shewhart–CUSUM quality control scheme in which the CUSUM limits help in detecting the smaller shifts whereas the Shewhart limits increase the sensitivity of the chart for the larger shifts. Similarly, the use of the combined Shewhart–EWMA scheme was recommended by Lucas and Saccucci[5] to make the EWMA chart more sensitive for the larger shifts. Another alternative was presented by Jiang *et al.*[6] to use the adaptive CUSUM procedure with EWMA-based shift estimators in which a range of shifts is targeted and the reference value of the CUSUM is updated using the EWMA estimate.

In this article, we propose the use of a mixed EWMA–CUSUM control chart with the motivation to further enhance the sensitivity of the control chart structure, particularly for the shifts of smaller magnitude in the process.

The organization of the rest of the article is as follows. In the next section, we present the basic design structure of CUSUM- and EWMA-type control charts. The details regarding the design structure of the proposed scheme are provided in Section 3. Section 4 consists of the comparison of the proposed scheme with its counterparts. An illustrative example of the proposed scheme is presented in Section 5, and finally, Section 6 summarizes the findings of this article.

[a]*Department of Statistics, Quaid-i-Azam University Islamabad, Islamabad, Pakistan*
[b]*Department of Mathematics and Statistics, King Fahad University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia*
[c]*Department of Quantitative Economics, IBIS UvA, University of Amsterdam, Plantage Muidergracht 12, 1018 TV, Amsterdam, The Netherlands*
**Correspondence to: Nasir Abbas, Department of Statistics, Quaid-i-Azam University Islamabad, Islamabad, Pakistan.*
[†]*E-mail: nasirabbas55@yahoo.com*

2. Classical CUSUM and EWMA charts

Shewhart-type control charts use only the current observation or sample to monitor the process. In this section, we consider control charts that also use previous observations along with the current observation. These mainly include CUSUM and EWMA schemes, and we provide here the details regarding their usual design structures (also known as classical CUSUM and EWMA control charts).

2.1. Classical CUSUM control charts

The CUSUM chart was originally introduced by Page[2] and is suited to detect small and sustained shifts in a process. These charts measure a cumulative deviation from the mean or a target value. The two versions of the CUSUM chart, used to evaluate an out of control condition, are the V-mask CUSUM and the tabular CUSUM. The V-mask procedure, which is not very common in use, normalizes the deviations from the mean and plots these deviations. As long as these deviations are plotted around the target value, the process is said to be in control, otherwise out of control. The tabular method of evaluating a CUSUM chart is commonly used and is similar to a Shewhart-type control chart. In this method, we plot a function of the subgroup average against the control limits, which are set according to a prefixed ARL_0 value. Execution of the tabular CUSUM scheme for controlling the location parameter of the process is performed using two statistics called C^+ and C^-, which are defined as

$$\begin{aligned} C_i^+ &= max\left[0, (X_i - \mu_0) - k + C_{i-1}^+\right] \\ C_i^- &= max\left[0, -(X_i - \mu_0) - k + C_{i-1}^+\right] \end{aligned} \tag{1}$$

where X_i denotes the ith observation (e.g. the sample mean of sample i), μ_0 is the target mean, and k is the reference value, which is usually chosen equal to half of the shift (in standard units) to be detected. The quantities C^+ and C^- are known as upper and lower CUSUM statistics, respectively, which are initially set to zero. These two statistics are plotted against the control limit h. As long as the values of C_i^+ and C_i^- are plotted inside the control limit h, the process is said to be in control, otherwise out of control. If the statistic C_i^+ is plotted above h, the process mean is said to be shifted above the target value, and if the statistic C_i^- is plotted above h, the process is said to be shifted below the target value. The quantities k and h are the parameters of the CUSUM control chart, and their proper selection is very important because it greatly influences the ARL performance of the CUSUM chart. Hawkins and Olwell[7] provided a complete ARL study for the tabular CUSUM charts for the mean of a normally distributed process with different choices of k and h. Some of their results are given in Table I, with $k = 0.5$ and δ representing the amount of shift (in standard units) in the process mean.

2.2. Classical time-varying EWMA control charts

The EWMA control chart was introduced by Roberts.[3] Like the CUSUM scheme, EWMA also uses the past information along with the current, but the weights attached to the data are exponentially decreasing as the observations become less recent. An EWMA control chart for monitoring the mean of a process is based on the statistic

$$Z_i = \lambda X_i + (1 - \lambda) Z_{i-1} \tag{2}$$

where i is the sample number and λ is the constant such that $0 < \lambda \leqslant 1$. The quantity Z_0 is the starting value, and it is taken equal to the target mean μ_0 or the average of the initial data in case when the information on the target mean is not available. The control structure of the EWMA chart (which includes the upper control limit (UCL), centre line (CL) and lower control limit (LCL)) is defined as

$$\left.\begin{aligned} \mathrm{LCL} &= \mu_0 - L\sigma_X \sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2i}\right)} \\ \mathrm{CL} &= \mu_0 \\ \mathrm{UCL} &= \mu_0 + L\sigma_X \sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2i}\right)} \end{aligned}\right\} \tag{3}$$

where σ_X is the standard deviation of the independent observations $\left(X_i{'}s\right)$, and L is the control limit coefficient that determines the width of the control limits. Like CUSUM charts, EWMA control charts also have two parameters (λ and L). λ determines the decline of weights, whereas L determines the width of the control limits, so jointly these two parameters determine the ARL performance of the EWMA charts. Steiner[8] provided a detailed study on the ARL performance of EWMA control chart with these

Table I. ARL values for the classical CUSUM scheme with $k = 0.5$

δ	0	0.25	0.5	0.75	1	1.5	2
$h = 4$	168	74.2	26.6	13.3	8.38	4.75	3.34
$h = 5$	465	139	38.0	17.0	10.4	5.75	4.01

time-varying limits used to monitor the mean of a normally distributed process. These ARLs are reproduced for $ARL_0 = 500$ and are given in Table II.

The results of Table II indicate that the smaller values of λ result in better ARL_1 performance for the smaller shifts but the larger values of λ provide more shelter against larger shifts. Now for a user of the EWMA chart, it is the balance of ARL_0 and ARL_1 that determines the parameters of the EWMA chart.

2.3. *Modifications of the classical CUSUM and EWMA control charts*

After the development of CUSUM and EWMA charts, several modifications of these charts have been presented to further enhance the performance of these charts. Lucas[4] presented the combined Shewhart–CUSUM quality control scheme in which Shewhart and CUSUM limits are used simultaneously. Lucas and Crosier[9] recommended the use of the fast initial response (FIR) CUSUM, which gives a head start to the CUSUM statistic by setting the initial values of the CUSUM statistic equal to some positive value (nonzero). This feature gives better ARL_1 performance but at the cost of a decrease in ARL_0. Yashchin[10] presented the weighted CUSUM scheme, which gives different weights to the previous information used in CUSUM statistic. Riaz *et al.*[11] applied the runs rules schemes to the CUSUM charts and showed that the runs rules–based CUSUM performs better than the classical CUSUM for small shifts. Similarly, on the EWMA side, Lucas and Saccucci[5] presented the combined Shewhart–EWMA quality control scheme, which gives better ARL_1 performance for both small and large shifts. Steiner[8] provided the FIR EWMA, which gives a head start to the initial value of the EWMA statistic (like FIR CUSUM) and hence improves the ARL_1 performance of the EWMA charts. Abbas *et al.*[12] applied the runs rules schemes to the EWMA charts and showed that the runs rules–based EWMA performs better than the classical EWMA for small shifts.

In the next section, we present a mixed EWMA–CUSUM quality control scheme for monitoring the mean of a normally distributed process. The inspiration is to get an improved ARL performance by combining the features of EWMA and CUSUM charts in a single control structure.

3. Design structure of the proposed mixed EWMA–CUSUM scheme

In this section, we propose an assortment of the classical EWMA and CUSUM schemes by combining the features of their design structures. The said proposal mainly depends on two statistics called M_i^+ and M_i^-, which are defined as

$$\left.\begin{aligned} M_i^+ &= \max\left[0, (Q_i - \mu_0) - a_i + M_{i-1}^+\right] \\ M_i^- &= \max\left[0, -(Q_i - \mu_0) - a_i + M_{i-1}^-\right] \end{aligned}\right\} \tag{4}$$

where a_i is a time-varying reference value for the proposed charting structure, and the quantities M_i^+ and M_i^- are known as the upper and lower CUSUM statistics, respectively, which are initially set to zero (i.e. $M_i^+ = M_i^- = 0$) and are based on the EWMA statistic Q_i, which is defined as

$$Q_i = \lambda_q Y_i + \left(1 - \lambda_q\right) Q_{i-1} \tag{5}$$

In Equation (5), λ_q is the constant like λ in Equation (2) such that $0 < \lambda_q \leqslant 1$ and the initial value of the Q_i statistic is set equal to the target mean, that is, $Q_0 = \mu_0$. Now the mean and variance of statistic Q_i is given as

$$\text{Mean}(Q_i) = \mu_0, \quad \text{Var}(Q_i) = \sigma_Y^2\left(\frac{\lambda}{2 - \lambda_q}\left(1 - \left(1 - \lambda_q\right)^{2i}\right)\right) \tag{6}$$

and this will be used later in the calculation of the parameters of the proposed chart. μ_0 and σ_Y are the population mean and standard deviation, respectively. If these two population parameters are unknown, these can be estimated from preliminary samples.

In Equations (4) and (5), we are considering the case of individual observations ($n = 1$), which may be extended easily for the subgroups. Now the statistics M_i^+ and M_i^- are plotted against the control limit, say b_i. As long as the values of M_i^+ and M_i^- are plotted inside the control limit, the process is said to be in control, otherwise out of control. It is to be noted here that if the statistic M_i^+ is plotted above b_i, the process mean is said to be shifted above the target value, and if the statistic M_i^- is plotted above b_i, the process is said to

Table II. ARL values for the classical EWMA scheme at $ARL_0 = 500$

δ	$\lambda = 0.1$, $L = 2.824$	$\lambda = 0.25$, $L = 3$	$\lambda = 0.5$, $L = 3.072$	$\lambda = 0.75$, $L = 3.088$
0	499.8921	500.8062	499.3591	499.3572
0.25	102.9906	169.4897	255.9586	321.2977
0.5	28.85911	47.4967	88.75299	139.8735
0.75	13.55984	19.22174	35.55055	62.46324
1	8.21866	10.3978	17.08736	30.57454
1.5	4.17073	4.76612	6.27401	9.7965
2	2.65838	2.93505	3.40149	4.46257

be shifted below the target value. The control limit b_i is selected according to a prefixed ARL_0. A large value of the prefixed ARL_0 will give a larger value of b_i and *vice versa*. The two quantities a_i and b_i are defined as

$$\left.\begin{aligned} a_i &= a^*\sqrt{Var(Q_i)} = a^*\sigma_Y\sqrt{\frac{\lambda}{2-\lambda_q}\left(1-(1-\lambda_q)^{2i}\right)} \\ b_i &= b^*\sqrt{Var(Q_i)} = b^*\sigma_Y\sqrt{\frac{\lambda}{2-\lambda_q}\left(1-(1-\lambda_q)^{2i}\right)} \end{aligned}\right\} \quad (7)$$

where a^* and b^* are the constants like k and h, respectively, in the classical set up for the CUSUM. The time-varying reference values a_i and b_i are due to the variance of the EWMA statistic in Equation (6). For a fixed value of a^*, we can select the value of b^* from the tables (that are given later in this section) that fix the ARL_0 at our desired level. In general, a_i is chosen equal to half of the shift (in units of the standard deviation of Q_i). Hence, we choose $a^*=0.5$ because it makes the CUSUM structure more sensitive to the small and moderate shifts (cf. Montgomery[1]), to which memory charts actually target.

To evaluate the ARL performance of a control scheme, there are different approaches used in the literature, including Markov chains, integral equations, Monte Carlo simulations and different types of approximations. We have used the Monte Carlo simulation approach in this article to evaluate the ARLs of the proposed chart. An algorithm in R language has been developed to calculate the run lengths. The algorithm is run 50,000 times to calculate the average of those 50,000 run length. A detailed study on the ARL performance of the proposed EWMA–CUSUM control chart to monitor the mean of a normally distributed process is provided in Tables III–V for some selective choices of δ, λ_q and b^*. For this purpose ARL_0s are fixed at 168, 400 and 500, which are the commonly used choices. For other values of ARL_0s one may easily obtain the results on similar lines.

Table III. ARL values for the proposed EWMA–CUSUM scheme with $a^*=0.5$ at $ARL_0=168$

δ	$\lambda_q=0.1$, $b^*=21.3$	$\lambda_q=0.25$, $b^*=13.29$	$\lambda_q=0.5$, $b^*=8.12$	$\lambda_q=0.75$, $b^*=5.48$
0	168.0441	168.0652	169.8763	171.0422
0.25	52.6449	54.1752	59.7829	68.15245
0.5	24.85945	22.40665	22.54895	24.12865
0.75	17.0208	14.0235	12.85555	12.60565
1	13.3323	10.4832	8.9565	8.2741
1.5	9.743	7.3272	5.78565	4.99665
2	7.90705	5.8231	4.4341	3.7365

Table IV. ARL values for the proposed EWMA–CUSUM scheme with $a^*=0.5$ at $ARL_0=400$

δ	$\lambda_q=0.1$, $b^*=33.54$	$\lambda_q=0.25$, $b^*=18.7$	$\lambda_q=0.5$, $b^*=10.52$	$\lambda_q=0.75$, $b^*=6.94$
0	402.0894	397.404	398.6486	400.8962
0.25	73.31955	78.02035	90.45915	108.0086
0.5	33.06085	29.0845	28.94885	31.44015
0.75	22.39445	17.79425	15.75695	15.66785
1	17.63975	13.2232	10.94695	10.17115
1.5	12.88105	9.113	6.9587	6.03875
2	10.45315	7.2235	5.2808	4.4203

Table V. ARL values for the proposed EWMA–CUSUM scheme with $a^*=0.5$ at $ARL_0=500$

δ	$\lambda_q=0.1$, $b^*=37.42$	$\lambda_q=0.25$, $b^*=20.18$	$\lambda_q=0.5$, $b^*=11.2$	$\lambda_q=0.75$, $b^*=7.32$
0	498.3882	502.018	507.9555	507.5152
0.25	80.13585	83.7529	100.2635	121.9883
0.5	35.524	30.88825	30.7466	33.5054
0.75	24.0522	18.8755	16.6399	16.5139
1	18.8637	13.8816	11.45835	10.6107
1.5	13.79075	9.6036	7.29565	6.3101
2	11.19775	7.59055	5.52345	4.589

The relative standard errors for the results provided in Tables III–V are also calculated and found to be less than 1.2%. Moreover, we have also replicated the ARL results of the classical CUSUM and the classical EWMA using our simulation algorithm and found almost similar results as by Hawkins and Olwell[7] and Lucas and Saccucci,[5] respectively, ensuring the validity of the simulation algorithm used.

The main findings about our proposed EWMA–CUSUM quality control scheme for monitoring the mean of a normally distributed process are given as follows:

1. Mixing the EWMA and CUSUM schemes really boosts the ARL performance of the resulting combination of the two charts especially for small and moderate shifts in the process (cf. Tables III–V).
2. For detecting small shifts in the process, the performance of the proposed scheme is better with smaller values of λ_q and *vice versa* (cf. Tables III–V).
3. The proposed scheme is ARL unbiased, that is, for a fixed value of ARL_0, the ARL_1 decreases with a decrease in the value of δ and *vice versa* (cf. Tables III–V).
4. For a fixed value of δ, the ARL_1 of the proposed scheme decreases with a decrease in ARL_0 (cf. Tables III–V).
5. For a fixed value of ARL_0, the control limit coefficient b^* decreases with the increase in λ_q (cf. Tables III–V).

4. Comparisons

In this section, we present a comprehensive comparison of the proposed mixed EWMA–CUSUM scheme with some existing representative EWMA and CUSUM control charts available in the literature. The performance of the control chart is compared in terms of ARL. The set of the schemes considered for the comparison consist of the classical CUSUM, the classical EWMA, the FIR CUSUM, the FIR EWMA, the adaptive CUSUM with EWMA-based shift estimator, the weighted CUSUM and the runs rules–based CUSUM and EWMA. The ARLs of these charts are given in Tables I, II and VI–XIII.

4.1. Proposed versus classical CUSUM: The ARL values for the classical CUSUM control scheme proposed by Page[2] are given in Table I. Comparison of the classical CUSUM with the proposed schemes reveals that the proposed scheme is performing really good for all the values of λ_q, particularly for the smaller values of λ_q. We can see that, for all the values of λ_q, the proposed scheme has better ARL performance for small shifts, that is, $\delta < 1$, as compared with the classical CUSUM (cf. Table I versus Table III).

4.2. Proposed versus classical time-varying EWMA: The ARL values for the classical EWMA with time-varying limits, given by Steiner,[8] are provided in Table II. Comparing the classical EWMA with the proposed scheme, we observed that the proposed scheme have better ARL_1 performance for small shifts, that is, $\delta < 0.75$, with its respective values of λ_q (cf. Table II versus Table V).

4.3. Proposed versus FIR CUSUM: The FIR CUSUM presented by Lucas and Crosier[9] provides a head start to the CUSUM statistic. The ARLs of the CUSUM with FIR feature are given in Table VI in which head start is represented by C_0. The FIR feature decreases the ARL_0 of the CUSUM chart, and more importantly this decreased ARL_0 becomes very small for the larger values of C_0 (for $C_0 = 2$, $ARL_0 = 149$),

Table VI. ARL values for the FIR CUSUM scheme with $k = 0.5$

δ	0	0.25	0.5	0.75	1	1.5	2
$h = 4$, $C_0 = 1$	163	71.1	24.4	11.6	7.04	3.85	2.7
$h = 5$, $C_0 = 2$	149	62.7	20.1	8.97	5.29	2.86	2.01

Table VII. ARL values for the FIR EWMA scheme and proposed chart

	FIR EWMA		EWMA–CUSUM	
	$\lambda = 0.1$, $L = 3$	$\lambda = 0.1$, $L = 3$	$\lambda_q = 0.1$, $a^* = 0.5$	$\lambda_q = 0.1$, $a^* = 0.5$
δ	$f = 0.4$	$f = 0.5$	$b^* = 37.94$	$b^* = 40.8$
0	515.6	613.8	516.48	613.62
0.25	83.1	99.2	81.03	85.79
0.5	18.5	22.1	35.76	37.55
0.75	7.3	8.8	24.2	25.41
1	3.8	4.6	19.01	19.94
1.5	1.7	2.1	13.9	14.55
2	1.3	1.4	11.29	11.8
3	1	1	8.48	8.88
4	1	1	6.96	7.29

Table VIII. ARL values for adaptive CUSUM with $\delta^{+}_{\min}$ and $\lambda = 0.3$ at $ARL_0 = 400$

δ	$\gamma = 1.5$ $h = 5.05$	$\gamma = 2$ $h = 4.73$	$\gamma = 2.5$ $h = 4.505$	$\gamma = 3$ $h = 4.394$	$\gamma = 4$ $h = 4.337$	$\gamma = \infty$ $h = 4.334$
0	399.7	400.85	400.19	399.29	399.39	399.97
0.25	92.82	91.65	88.96	87.02	85.81	85.8
0.5	30.52	30.1	29.32	28.79	28.46	28.45
0.75	14.7	14.5	14.2	14	13.89	13.88
1	9.07	8.96	8.81	8.72	8.67	8.66
1.5	4.89	4.87	4.84	4.83	4.83	4.82
2	3.23	3.25	3.28	3.31	3.35	3.34

Table IX. ARL values for the symmetric two-sided weighted CUSUM scheme at $ARL_0 = 500$

$k = 0.5$		δ			
γ	h	0.5	1	1.5	2
0.7	3.16	86.30	15.90	6.08	3.52
0.8	3.46	70.20	13.30	5.66	3.50
0.9	3.97	54.40	11.40	5.50	3.60
1.0	5.09	39.00	10.50	5.81	4.02

Table X. WL, AL and ARL values for the runs rules–based CUSUM scheme 1 at $ARL_0 = 168$

Limits		δ						
WL	AL	0	0.25	0.5	0.75	1	1.5	2
3.42	4.8	168	71.8715	25.5644	13.5392	8.6598	5.0776	3.6786
3.44	4.6	168	72.258	25.6532	13.5	8.5682	5.0128	3.6072
3.48	4.4	168	71.936	25.5934	13.4956	8.516	4.936	3.5246
3.53	4.2	168	71.399	25.3002	13.3322	8.4044	4.8282	3.423

Table XI. WL, AL and ARL values for the runs rules–based CUSUM scheme 2 at $ARL_0 = 168$

Limits		δ						
WL	AL	0	0.25	0.5	0.75	1	1.5	2
3.5	4.44	168	71.489	25.3786	13.3984	8.462	4.9412	3.5406
3.6	4.19	168	72.938	25.3676	13.3524	8.3828	4.83	3.424
3.7	4.08	168	73.1095	25.3692	13.3058	8.3442	4.7772	3.3762
3.8	4.03	168	73.589	25.4026	13.2766	8.3156	4.75	3.3474

Table XII. ARL values for the runs rules based EWMA scheme 1 at $ARL_0 = 500$

δ	$\lambda = 0.1$ $L_s = 2.556$	$\lambda = 0.25$ $L_s = 2.554$	$\lambda = 0.5$ $L_s = 2.36$	$\lambda = 0.75$ $L_s = 2.115$
0	501.7558	505.5284	501.2598	502.0725
0.25	103.3109	169.1349	235.1138	280.6187
0.5	29.5748	47.0105	78.0771	108.8792
0.75	14.3216	19.2776	30.8742	45.3405
1	8.9561	10.5964	15.1992	22.1033
1.5	4.9197	5.2578	6.1014	7.7862
2	3.4498	3.5527	3.6815	4.0883

which is not recommended in case of sensitive processes like in health care (cf. Bonetti *et al.*[13]). Comparing the proposed scheme with the FIR CUSUM, we found that for smaller values of λ_q, the proposed scheme has a better ARL performance for small shifts than the FIR CUSUM, even if the FIR CUSUM does not have the fixed ARL_0 at 168 but has smaller ARL_0 value, that is, 149 (cf. Table VI versus Table III).

Table XIII. ARL values for the runs rules based EWMA scheme 2 at $ARL_0 = 500$

δ	$\lambda = 0.1$ $L_s = 2.3$	$\lambda = 0.25$ $L_s = 2.345$	$\lambda = 0.5$ $L_s = 2.202$	$\lambda = 0.75$ $L_s = 1.982$
0	502.883	499.6153	505.3564	501.9698
0.25	66.6864	97.0108	133.7117	155.7078
0.5	21.4251	31.2023	46.3541	57.7739
0.75	11.7427	14.4295	20.6223	26.0312
1	7.5539	8.6761	11.0991	13.8363
1.5	4.4676	4.7066	5.1336	5.7812
2	3.4534	3.549	3.6276	3.7787

4.4. Proposed versus FIR EWMA: FIR EWMA presented by Steiner[8] is similar to the FIR CUSUM as it also gives a head start to the EWMA statistic. The control limits for the FIR-based EWMA chart are given as

$$\mu_0 \pm L\sigma_X\left(1-(1-f)^{1+a(i-1)}\right)\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2i}\right)}$$

where $a = (-2/\log(1-f) - 1)/19$. The ARLs for the FIR EWMA with $\lambda = 0.1$ and the proposed chart with $\lambda_q = 0.1$ are given in Table VII. Comparing the proposed scheme with the FIR EWMA, we found that the proposed scheme has a better performance than the FIR EWMA for smaller shifts, that is, $\delta < 0.5$. For moderate and larger shifts, FIR EWMA seems superior as compared with the proposed chart.

4.5. Proposed versus adaptive CUSUM with EWMA-based shift estimator: Jiang *et al.*[6] proposed the use of adaptive CUSUM with EWMA-based shift estimator. They used the concept of adaptively updating the reference value of the CUSUM chart using the EWMA estimator and then using a suitable weighting function. The ARL values for the adaptive CUSUM are given in Table VIII in which δ^+_{min}, λ, γ and h are the parameters of the chart. Comparing the performance of the proposed scheme, we noticed that the proposed scheme is outperforming the adaptive CUSUM for small values of δ. For moderate and large values of δ, both the proposed scheme and the adaptive CUSUM have almost the same ARL performance (cf. Table VIII versus Table IV).

There is also an adaptive EWMA chart (cf. Capizzi and Masarotto[14]), but its performance is inferior to the adaptive CUSUM, so the results of our proposal are superior to the adaptive EWMA as well.

4.6. Proposed versus weighted CUSUM: Weighted CUSUM presented by Yashchin[10] gives weights to the past information in the CUSUM statistic. The ARLs for the weighted CUSUM are given in Table IX in which the weights given to the past information are represented by γ. The comparison of the proposed scheme with the weighted CUSUM shows that the proposed scheme is performing better than the weighted CUSUM for the small and moderate shifts (e.g. $\delta < 1.5$). For larger values of δ, the weighted CUSUM almost coincide with the proposed scheme (cf. Table IX versus Table V).

4.7. Proposed versus runs rules–based CUSUM: Riaz *et al.*[11] proposed the use of the runs rules schemes with the design structure of the CUSUM charts. The ARLs for the two runs rules–based CUSUMs are given in Tables X and XI, in which WL and AL are representing the warning limits and action limits, respectively. The comparison of the proposed scheme with both the runs rules–based CUSUM schemes shows that the proposed scheme has the ability to perform better than the runs rules–based CUSUM for all the choices of λ_q (cf. Tables X and XI versus Table III).

4.8. Proposed versus runs rules–based EWMA: Abbas *et al.*[12] proposed the use of the runs rules schemes EWMA structure. The ARLs for the two runs rules–based EWMAs are given in Tables XII and XIII. The comparison of the proposed schemes with both the runs rules–based EWMA schemes shows that the proposed scheme is performing better as long as $\lambda > 0.1$ for the runs rules–based EWMA schemes. For $\lambda = 0.1$, runs rules–based EWMA scheme 2 becomes a bit superior to the proposed chart (cf. Tables XII and XIII versus Table V).

4.9. Overall view: To provide an overall comparative view of the proposed scheme with the other existing counterparts, we have made some graphical displays in the form of ARL curves. Three selective graphs of different charts/schemes (discussed in Tables I–V and VI–XIII) are given in Figures 1–3. In Figures 1–3, RR CUSUM (EWMA) stands for the runs rules–based CUSUM (EWMA) schemes and the other terms/symbols used are self-explanatory. By examining the graphs of ARL curves of different schemes under study, we found that the ARL curves of the proposed schemes are on the lower side, which shows evidence for the domination of the proposed scheme over the other schemes. For the smaller values of δ, the difference between the ARL of the proposed scheme and the other schemes is larger, whereas for the moderate values of δ, this difference almost disappears. For larger values of δ, the ARL curve of the proposed chart seems above the ARL curves of some other charts, showing the poor performance of the proposed chart for larger shifts.

To sum up, we may infer that in general the proposed scheme is good to detect small and moderate shifts whereas for larger shifts its performance is inferior to some of the other schemes under investigation.

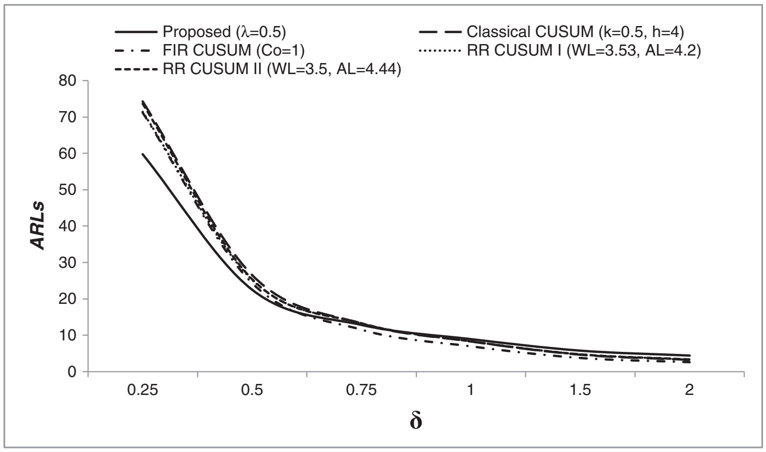

Figure 1. ARL curves for the proposed scheme, the classical CUSUM, the FIR CUSUM and the runs rules–based CUSUMs at $ARL_0 = 168$

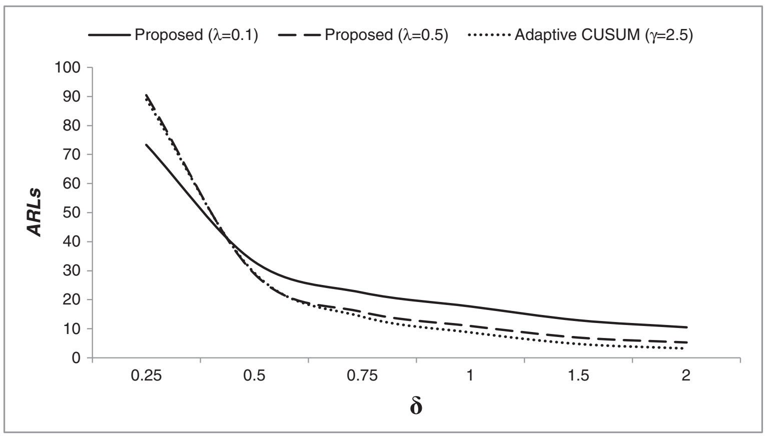

Figure 2. ARL curves for the proposed scheme and adaptive CUSUM at $ARL_0 = 400$

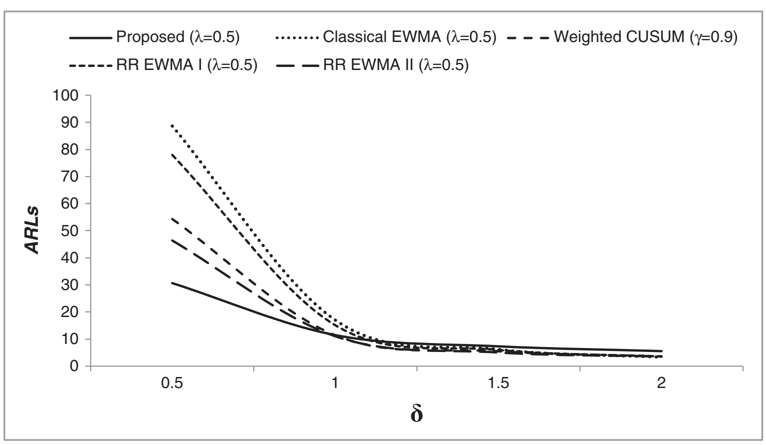

Figure 3. ARL curves for the proposed scheme, the classical EWMA, the FIR EWMA, the runs rules–based EWMA and the weighted CUSUM at $ARL_0 = 500$

5. Illustrative example

Besides exploring the statistical properties of a method, it is always good to provide its application on some data for illustration purposes. Authors generally provide these types of examples using real or simulated data sets, for example, Lucas and Crosier,[9] Khoo,[15] Antzoulakos and Rakitzis[16] and Riaz *et al.*[11] Following this inspiration, we present here an illustrative example to show how the

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Table XIV. Application example of the proposed scheme using $\lambda_q = 0.25$, $a^* = 0.5$ and $b^* = 20.18$ at $ARL_0 = 500$

Sample no.	$X_i = Y_i$	Q_i	a_i	M_i^+	M_i^-	b	Sample no.	$X_i = Y_i$	Q_i	a_i	M_i^+	M_i^-	b
1	−0.113	−0.028	0.125	0	0	5.045	21	0.781	0.452	0.189	3.175	0	7.627
2	−1.906	−0.498	0.156	0	0.341	6.306	22	−0.016	0.335	0.189	3.321	0	7.627
3	−1.891	−0.846	0.171	0	1.016	6.915	23	−0.061	0.236	0.189	3.368	0	7.627
4	0.508	−0.508	0.179	0	1.344	7.235	24	0.332	0.26	0.189	3.439	0	7.627
5	1.374	−0.037	0.184	0	1.198	7.409	25	1.391	0.543	0.189	3.793	0	7.627
6	0.05	−0.015	0.186	0	1.027	7.506	26	1.89	0.879	0.189	4.483	0	7.627
7	0.401	0.089	0.187	0	0.751	7.559	27	0.709	0.837	0.189	5.131	0	7.627
8	0.692	0.239	0.188	0.051	0.323	7.589	28	−0.82	0.423	0.189	5.364	0	7.627
9	0.851	0.392	0.188	0.255	0	7.606	29	1.481	0.687	0.189	5.863	0	7.627
10	0.927	0.526	0.189	0.593	0	7.615	30	0.314	0.594	0.189	6.268	0	7.627
11	2.187	0.941	0.189	1.346	0	7.621	31	2.231	1.003	0.189	7.082	0	7.627
12	0.02	0.711	0.189	1.868	0	7.623	32	0.802	0.953	0.189	7.846[a]	0	7.627
13	0.12	0.563	0.189	2.242	0	7.625	33	−1.25	0.402	0.189	8.059[a]	0	7.627
14	2.138	0.957	0.189	3.01	0	7.626	34	0.351	0.389	0.189	8.26[a]	0	7.627
15	0.183	0.764	0.189	3.585	0	7.627	35	1.362	0.632	0.189	8.703[a]	0	7.627
16	−2.389	−0.024	0.189	3.371	0	7.627	36	−0.529	0.342	0.189	8.856[a]	0	7.627
17	−0.269	−0.086	0.189	3.097	0	7.627	37	2.59	0.904	0.189	9.571[a]	0	7.627
18	0.317	0.015	0.189	2.923	0	7.627	38	0.287	0.75	0.189	10.132[a]	0	7.627
19	0.055	0.025	0.189	2.759	0	7.627	39	1.676	0.981	0.189	10.924[a]	0	7.627
20	1.293	0.342	0.189	2.912	0	7.627	40	−0.303	0.66	0.189	11.395[a]	0	7.627

[a]Proposed scheme giving out-of-control signal.

proposed scheme can be applied in the real situation. For this purpose, a data set is generated containing 40 observations. The first 20 observations are generated from the in-control situation (i.e. $N(0, 1)$ so that the target mean is 0), and the remaining 20 observations are generated from an out-of-control situation with a small shift introduced in the process (i.e. $N(0.5, 1)$). The classical CUSUM, the classical EWMA and the proposed scheme are applied to this data set, and the parameters are selected to be $k=0.5$ and $h=5.09$ for the classical CUSUM scheme, $\lambda=0.25$ and $L=2.998$ for the classical EWMA scheme, and $\lambda_q=0.25$, $a^*=0.5$ and $b^*=20.18$ for the proposed scheme to guarantee that $ARL_0=500$. The calculations for the proposed scheme are given in Table XIV, and the graphical display of all three control structures are provided in Figures 4–6, with the statistics C_i^+ and C_i^- plotted against the control limit h for

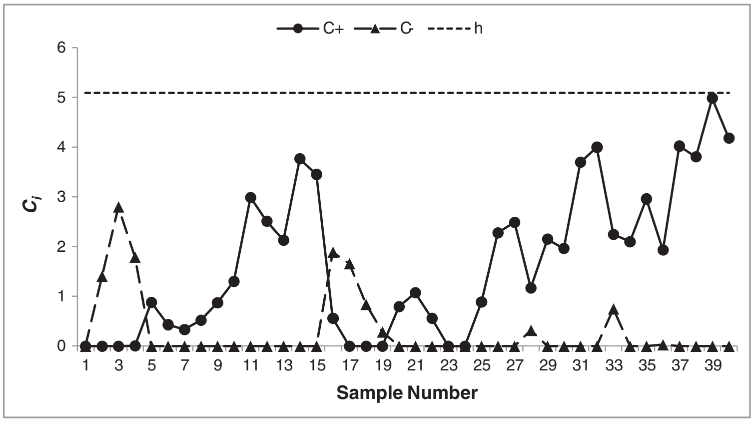

Figure 4. The classical CUSUM chart for the simulated data set using $k=0.5$ and $h=5.09$ at $ARL_0=500$

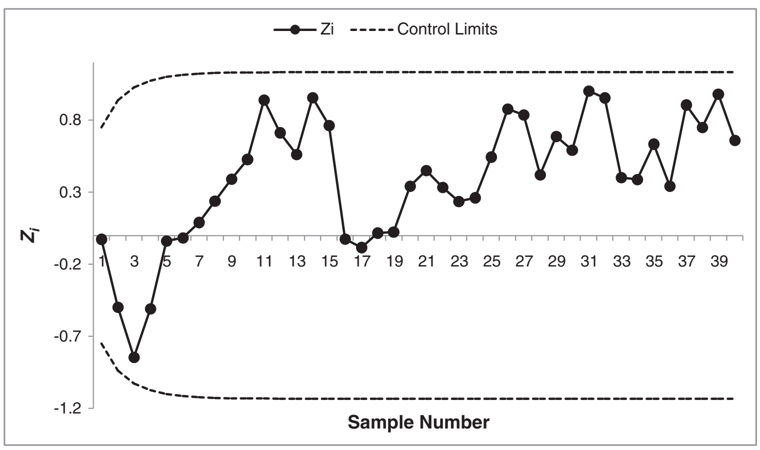

Figure 5. The classical EWMA chart for the simulated data set using $\lambda=0.25$ and $L=2.998$ at $ARL_0=500$

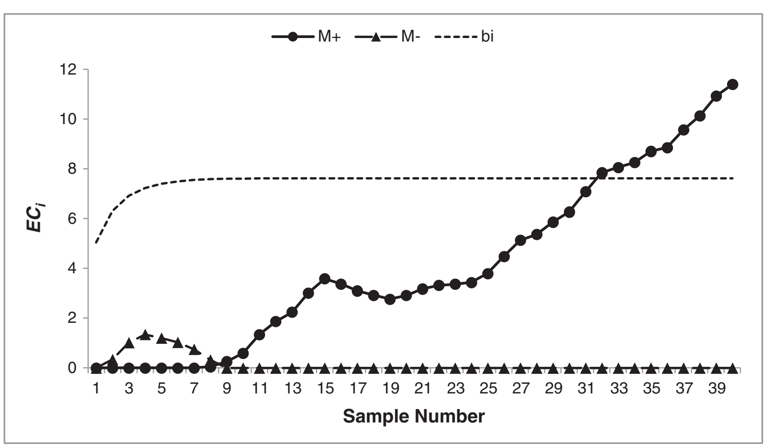

Figure 6. The proposed scheme for the simulated data set using $\lambda_q=0.25$, $a^*=0.5$ and $b^*=20.18$ at $ARL_0=500$

the classical CUSUM scheme, Z_i plotted against the control limits $\pm L$ for the classical EWMA scheme and M_i^+ and M_i^- plotted against the control limit b_i for the proposed scheme.

From Tables XIV and Figure 6, it is obvious that out-of-control signals are received at samples 32, 33, 34, 35, 36, 37, 38, 39 and 40 by the proposed scheme (giving eight out-of-control signals). Figures 4 and 5 show that the separate applications of the classical CUSUM and EWMA schemes fail to detect any out-of-control situation for the given data set. This clearly indicates superiority of the proposed scheme over the classical CUSUM and EWMA schemes, and it is exactly in accordance with the findings of Section 4.

6. Summary and conclusions

Control charts are widely used in processes to detect any special cause variations. Mainly, they are categorized into memory control charts and memoryless control charts. Memoryless control charts (like Shewhart control charts) are designed to detect the larger shifts in the process, whereas the design structure of the memory control charts is made such that they are good at detecting the small and moderate shifts in the process. CUSUM control charts and EWMA control charts are the two most commonly used memory control charts in the literature. These control schemes do not only use the current observation but also accumulate the information from the past to give a quick signal if the process is slightly off-target. In this article, we have combined the CUSUM and EWMA control schemes into a single control structure and proposed a mixed EWMA–CUSUM control scheme. The performance of the proposed scheme is compared with other CUSUM- and EWMA-type control charts, which are meant to detect small and moderate shifts in the process. The comparisons revealed that the proposed scheme is really good at detecting the smaller shifts in the process as compared with the other schemes under study.

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REFERENCES

1. Montgomery DC. Introduction to Statistical Quality Control. 6th ed. John Wiley & Sons: New York, 2009.
2. Page ES. Continuous Inspection Schemes. *Biometrika* 1954; **41**:100–115.
3. Roberts SW. Control Chart Tests Based on Geometric Moving Averages. *Technometrics* 1959; **1**:239–250.
4. Lucas JM. Combined Shewhart–CUSUM Quality Control Schemes. *Journal of Quality Technology* 1982; **14**(2): 51–59.
5. Lucas JM, Saccucci MS. Exponentially Weighted Moving Average Control Schemes: Properties and Enhancements. *Technometrics* 1990; **32**:1–12.
6. Jiang W, Shu L, Aplet DW. Adaptive CUSUM Procedures with EWMA-Based Shift Estimators. *IIE Transactions* 2008; **40**(10):992–1003.
7. Hawkins DM, Olwell DH. Cumulative Sum Charts and Charting Improvement. Springer: New York, 1998.
8. Steiner SH. EWMA control charts with time varying control limits and fast initial response. *Journal of Quality Technology* 1999; **31**(1):75–86.
9. Lucas JM, Crosier RB. Fast Initial Response for CUSUM Quality-Control Scheme. *Technometrics* 1982; **24**:199–205.
10. Yashchin E. Weighted Cumulative Sum Technique. *Technometrics* 1989; **31**(1):321–338.
11. Riaz M, Abbas N, Does RJMM. Improving the Performance of CUSUM Charts. *Quality and Reliability Engineering International* 2011; **27**(4):415–424.
12. Abbas N, Riaz M, Does RJMM. Enhancing the Performance of EWMA Charts. *Quality and Reliability Engineering International* 2010; **27**(6):821–833.
13. Bonetti PO, Waeckerlin A, Schuepfer G, Frutiger A. Improving Time-Sensitive Processes in the Intensive Care Unit: the Example of 'Door-to-Needle Time' in Acute Myocardial Infarction. *International Journal for Quality in Health Care* 2000; **12**(4):311–317.
14. Capizzi G, Masarotto G. An Adaptive Exponentially Weighted Moving Average Control Chart. *Technometrics* 2003; **45**(3):199–207.
15. Khoo MBC. Design of Runs Rules Schemes. *Quality Engineering* 2004; **16**:27–43.
16. Antzoulakos DL, Rakitzis AC. The Modified *r* out of *m* Control Chart. *Communication in Statistics-Simulations and Computations* 2008; **37**(2):396–408.

Authors' biographies

Nasir Abbas got his M.Sc. in Statistics from the Department of Statistics Quaid-i-Azam University Islamabad Pakistan in 2009; M.Phil in Statistics from the Department of statistics Quaid-i-Azam University Islamabad Pakistan in 2011 and currently he is pursuing his Ph.D. in Statistics from the Institute if Business and Industrial Statistics University of Amsterdam The Netherlands. Additionally, he is also serving as Assistant Census Commissioner in Pakistan Bureau of Statistics. His current research interests include Quality Control particularly control charting methodologies. His e-mail address is: nasirabbas55@yahoo.com

Muhammad Riaz earned his B.Sc. with Statistics and Mathematics as major subjects from the Government Gordon College Rawalpindi, University of the Punjab Lahore Pakistan in 1998; M.Sc. in Statistics from the Department of Mathematics and Statistics Quaid-i-Azam University Islamabad Pakistan in 2001; M.Phil in Statistics from the Department of Mathematics and statistics, Allama Iqbal Open University Islamabad Pakistan in 2006 and Ph.D. in Statistics from the Institute if Business and Industrial Statistics University of Amsterdam The Netherlands in 2008. He served as a Statistical Officer in the Ministry of Food, Agriculture and Livestock, Islamabad, Pakistan during 2002–2003, as a Staff Demographer in the Pakistan Institute of Development Economics, Islamabad, Pakistan during 2003–2004, as a Lecturer in the Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan during 2004–2007, as an Assistant Professor in the Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan during 2007–2010. He is serving as an Assistant Professor in the Department of Mathematics and Statistics, King Fahad University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia from 2010-Present. His current research interests include Statistical Process Control, Non-Parametric techniques and Experimental Designs. His e-mail contacts are: riaz76qau@yahoo.com

Ronald J.M.M. Does graduated (PhD 1982) in mathematical statistics from the University of Leiden. Currently, he is professor of industrial statistics at the University of Amsterdam, managing director of the Institute for Business and Industrial Statistics and director of the Graduate School of Executive Programmes at the Amsterdam Business School. His research activities are the design of control charts for nonstandard situations, the methodology of Lean Six Sigma and healthcare engineering.