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CS-EWMA Chart for Monitoring Process Dispersion

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Control charts are the most extensively used technique to detect the presence of special cause variations in processes. They can be classified into memory and memoryless control charts. Cumulative sum and exponentially weighted moving average control charts are memory-type control charts as their control structures are developed in such a way that the past information is not ignored as it is done in the case of memoryless control charts, like the Shewhart-type control charts. The present study is based on the proposal of a new memory-type control chart for process dispersion. This chart is named as CS-EWMA chart as its plotting statistic is based on a cumulative sum of the exponentially weighted moving averages. Comparisons with other memory charts used to monitor the process dispersion are done by means of the average run length. An illustration of the proposed technique is done by applying the CS-EWMA chart on a simulated dataset. Copyright © 2012 John Wiley & Sons, Ltd.

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1. Introduction

The control chart is a technique to detect the presence of special cause variations in a process. It started with Shewhart control charts containing the average $\bar{X}$ chart for process location and the range R, standard deviation S, and variance S^2 charts for process dispersion. These charts were later classified as memoryless control charts because their control structure (containing the plotting statistic and the decision rule) is merely based on the last observation. This makes these charts relatively less sensitive to small shifts in the process parameter(s), which has led to the proposal of memory-type control charts, like cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts. These charts are designed such that their plotting statistic makes use of both past and present information. Due to this fact, the performance of memory charts (generally measured in terms of the average run length (*ARL*), defined as the expected number of subgroups required to detect a change in the process parameter) is better for small and moderate shifts as compared with the memoryless control charts.

Page[1] introduced the CUSUM chart for monitoring the increase in process dispersion using sample ranges. Following him, Tuprah and Ncube,[2] Chang and Gan,[3] and Acosta-Mejia *et al.*[4] proposed several improved versions of the CUSUM chart for process dispersion. On the other hand, Wortham and Ringer[5] suggested an EWMA control chart for monitoring the process dispersion. Ng and Case[6] and Crowder and Hamilton[7] proposed improved versions of the EWMA chart for monitoring process variance. Castagliola[8] and Castagliola *et al.*[9] proposed EWMA and CUSUM, respectively, control charts for monitoring the process variance based on a logarithmic transformation of the sample variance. This article proposes a new memory-type control chart based on the same transformation, named as CS-EWMA chart, for monitoring the process dispersion by mixing the effects of EWMA and CUSUM charts.

After presenting the basic structures of the EWMA and CUSUM charts for monitoring the process dispersion in the next section, we present the details of our proposed CS-EWMA chart for process standard deviation in the subsequent section. Then, in Section 4, we compare its performance with some existing EWMA-type and CUSUM-type charts for process dispersion. Section 5 contains an illustrative example, and finally Section 6 contains a summary and the conclusions of this article.

2. EWMA and CUSUM charts for process dispersion

2.1. S^2-EWMA control chart

Castagliola[8] proposed an S^2-EWMA control chart for monitoring the process dispersion. This control structure is based on a three-parameter logarithmic transformation which is given as

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$$T_j = a + b\ln\left(S_j^2 + c\right) \tag{1}$$

where S_j^2 is the sample variance for jth sample defined as $S_j^2 = \frac{\sum_{i=1}^{n}(X_{ij}-\bar{X}_j)^2}{n-1}$, X_{ij} represents the ith observation from the jth sample of size n from a normal distribution with mean μ, standard deviation σ_0, and $\bar{X}_j$ is the average of the jth sample. The constants a, b, and c are defined as $b = B(n)$, $c = C(n)\sigma_0^2$, and $a = A(n) - 2B(n)\ln(\sigma_0)$ as in the work of Castagliola.[8] He derived the distribution of T_j and showed that if the constants a, b, and c are judiciously selected, then the distribution of variable T_j becomes very close to the normal distribution with mean $\mu_T(n)$ and variance $\sigma_T^2(n)$, i.e. $T_j \approx N\left(\mu_T(n), \sigma_T^2(n)\right)$ (cf. Appendix A in Castagliola[8]). Table I reproduces the values of $A(n)$, $B(n)$, $C(n)$, $\mu_T(n)$, and $\sigma_T(n)$ for $n = 3, 4, 5, \ldots, 15$ from Table I in the work of Castagliola.[8]

Now, using the approximately normally distributed variable T_j from Castagliola,[8] the plotting statistic for the S^2-EWMA chart is defined as

$$Z_j = \lambda T_j + (1-\lambda)Z_{j-1} \tag{2}$$

where λ is the sensitivity parameter chosen as $0 < \lambda \leqslant 1$, and the initial value of Z_j is taken as $Z_0 = A(n) + B(n)\ln(1 + C(n))$. The control limits for the statistic given in Equation (2) are given as (cf. Lucas and Saccucci[10])

$$LCL = \mu_T(n) - L\sqrt{\frac{\lambda}{2-\lambda}}\sigma_T(n), \qquad CL = \mu_T(n), \qquad UCL = \mu_T(n) + L\sqrt{\frac{\lambda}{2-\lambda}}\sigma_T(n) \tag{3}$$

where L is the control limit coefficient that determines the distance between LCL and UCL. The ARL values of S^2-EWMA are given in Table II for different values of λ, where τ represents the amount of shift in the standard deviation (i.e. $\tau = {}^{\sigma_1}/_{\sigma_0}$, with σ_1 representing the shifted standard deviation), and the in-control ARL (represented by ARL_0) is fixed at 200, as is done in earlier work on this subject. The ARLs in this article are evaluated through simulation procedures by running 10^5 replications. The program is developed in R language and available from the authors on request. The out-of-control ARL is represented by ARL_1 throughout the article.

Table I. Values of $\mu_T(n)$, $\sigma_T(n)$, $A(n)$, $B(n)$ and $C(n)$

n	$A(n)$	$B(n)$	$C(n)$	$\mu_T(n)$	$\sigma_T(n)$
3	−0.6627	1.8136	0.6777	0.02472	0.9165
4	−0.7882	2.1089	0.6261	0.01266	0.9502
5	−0.8969	2.3647	0.5979	0.00748	0.9670
6	−0.9940	2.5941	0.5801	0.00485	0.9765
7	−1.0827	2.8042	0.5678	0.00335	0.9825
8	−1.1647	2.9992	0.5588	0.00243	0.9864
9	−1.2413	3.1820	0.5519	0.00182	0.9892
10	−1.3135	3.3548	0.5465	0.00141	0.9912
11	−1.3820	3.5189	0.5421	0.00112	0.9927
12	−1.4473	3.6757	0.5384	0.00090	0.9938
13	−1.5097	3.8260	0.5354	0.00074	0.9947
14	−1.5697	3.9705	0.5327	0.00062	0.9955
15	−1.6275	4.1100	0.5305	0.00052	0.9960

Table II. *ARL* values for the S^2-EWMA chart with $n = 5$ and $ARL_0 = 200$

τ	$\lambda = 0.05$ $L = 2.269$	$\lambda = 0.1$ $L = 2.452$	$\lambda = 0.2$ $L = 2.592$	$\lambda = 0.3$ $L = 2.634$	$\lambda = 0.4$ $L = 2.643$	$\lambda = 0.5$ $L = 2.639$
0.5	9.257	6.866	5.616	5.448	5.928	7.606
0.6	11.679	8.931	7.856	8.481	10.735	16.909
0.7	16.101	13.03	13.064	16.699	25.58	48.763
0.8	26.108	23.679	29.961	46.058	80.157	166.467
0.9	63.459	70.501	107.839	169.543	274.071	474.331
0.95	133.554	153.817	204.856	264.011	327.699	392.103
1	199.781	200.702	200.756	200.262	200.59	199.224
1.05	78.77	92.938	98.675	98.004	97.342	96.541
1.1	32.542	41.746	47.688	49.736	50.94	51.373
1.2	11.986	15.382	17.449	18.537	19.274	20.022
1.3	7.064	8.766	9.571	9.909	10.226	10.527
1.4	5.054	6.09	6.419	6.517	6.6	6.753
1.5	3.983	4.722	4.835	4.801	4.787	4.82
2	2.133	2.418	2.343	2.225	2.148	2.098
3	1.338	1.456	1.395	1.336	1.294	1.267

Note that the results from Table II coincide with the results of Table III of Castagliola.[8]

2.2. *CUSUM-S^2* control chart

Taking inspiration from the S^2-EWMA chart, Castagliola *et al.*[9] proposed a CUSUM-S^2 chart for monitoring the process dispersion, which is based on the statistic T_j given in Equation (1). The CUSUM-S^2 chart uses two plotting statistics, named as C^+ and C^-, and given as

$$\left.\begin{aligned} C_j^+ &= \max\left[0, \left(T_j - \mu_T(n)\right) - K + C_{j-1}^+\right] \\ C_j^- &= \max\left[0, -\left(T_j - \mu_T(n)\right) - K + C_{j-1}^-\right] \end{aligned}\right\} \tag{4}$$

where K is the reference value and the sensitivity parameter of the CUSUM-S^2 chart. The initial values for the plotting statistics given in Equation (4) are taken equal to zero, i.e. $C_0^+ = C_0^- = 0$. These plotting statistics are plotted against a control limit H, and an out-of-control signal is received if either of the two statistics (i.e. C^+ and C^-) is plotted above H. K and H are jointly the two parameters of CUSUM-S^2 chart, and their standard forms are given as

$$K = k\sigma_T(n), \qquad H = h\sigma_T(n) \tag{5}$$

where k and h are constants which determine the properties of the chart. The *ARL* values for the CUSUM-S^2 chart for different choices of its parameters are given in Table III, with ARL_0 fixed at 200.

3. Proposed CS-EWMA chart

In this section, we propose a memory-type control chart which is based on mixing the effects of EWMA and CUSUM charts into a single control chart structure. For the location parameter, this idea was explored by Abbas *et al.*[11] Again, let X_{ij} (i.e. the ith observation of jth sample with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots$) be distributed normally with mean μ and variance σ_0^2 under an in-control situation. Then, the two plotting statistics (named as M_j^+ and M_j^-) for the proposed CS-EWMA chart are given as

$$\left.\begin{aligned} M_j^+ &= \max\left[0, \left(Q_j - \mu_T(n)\right) - K'_q + M_{j-1}^+\right] \\ M_j^- &= \max\left[0, -\left(Q_j - \mu_T(n)\right) - K'_q + M_{j-1}^-\right] \end{aligned}\right\} \tag{6}$$

where K'_q is the reference value for the proposed chart, like K in Equation (4). The initial value for both plotting statistics is taken equal to zero, i.e. $M_0^+ = M_0^+ = 0$. Q_j is the EWMA statistic which is defined as

$$Q_j = \lambda_q T_j + \left(1 - \lambda_q\right) Q_{j-1} \tag{7}$$

where λ_q is the smoothing constant like λ in Equation (2) and is chosen as $0 < \lambda_q \leqslant 1$. T_j is the statistic defined in Equation (1). The initial value for the statistic Q_j is taken as $Q_0 = A(n) + B(n)\ln(1 + C(n))$. The statistics M_j^+ and M_j^- are now plotted against the control limit H'_q, and an out-of-control signal is received if either of the two plotting statistics given in Equation (6) is plotted above H'_q. If M_j^+ is

Table III. *ARL* values for the CUSUM-S^2 chart with $n = 5$ and $ARL_0 = 200$

τ	$K=0.1$ $H=10.53$	$K=0.25$ $H=6.476$	$K=0.5$ $H=3.855$	$K=0.75$ $H=2.62$	$K=1$ $H=1.906$
0.5	9.059	6.516	5.199	5.071	6.121
0.6	11.382	8.452	7.303	8.264	13.223
0.7	15.489	12.169	12.295	18.314	43.032
0.8	24.394	21.619	29.699	60.662	170.474
0.9	54.649	63.997	116.766	226.253	475.443
0.95	114.332	143.17	213.662	294.554	379.161
1	199.846	199.241	199.841	200.769	199.625
1.05	102.864	104.01	104.806	102.974	100.707
1.1	52.641	50.675	53.502	54.227	54.99
1.2	25.057	20.867	20.373	20.777	21.353
1.3	16.451	12.763	11.255	11.092	11.18
1.4	12.434	9.256	7.654	7.2	7.115
1.5	10.068	7.341	5.832	5.26	5.126
2	5.509	3.885	2.873	2.41	2.197
3	3.347	2.379	1.686	1.404	1.3

plotted above H'_q, that would indicate a positive shift in the process standard deviation, and if the value of M_j^- gets larger than H'_q, then it would be declared that the process standard deviation has shifted downwards. The standard forms of K'_q and H'_q depending upon the variance of Q_j (i.e. variance $\left(Q_j\right) = \sigma_T^2(n)\left(\frac{\lambda_q}{2-\lambda_q}\right)$, cf. Equation (3)) are given as

$$\left.\begin{aligned} K'_q &= k_q\left(\sigma_T(n)\sqrt{\frac{\lambda_q}{2-\lambda_q}}\right) = K_q\sqrt{\frac{\lambda_q}{2-\lambda_q}} \\ H'_q &= h_q\left(\sigma_T(n)\sqrt{\frac{\lambda_q}{2-\lambda_q}}\right) = H_q\sqrt{\frac{\lambda_q}{2-\lambda_q}} \end{aligned}\right\} \tag{8}$$

where $K_q = k_q\sigma_T(n)$ and $H_q = h_q\sigma_T(n)$. Here, in Equation (8), we have used the asymptotic standard deviation of the statistic Q, but the practitioner may use the exact standard deviation as discussed by Steiner.[12] Note that the CUSUM-S^2 is a special case of the proposed CS-EWMA chart with $\lambda_q = 1$. Finally, a detailed study on the *ARL* performance of the proposed CS-EWMA chart is given in Tables IV–IX, where the ARL_0 is fixed at 200 and $n = 5$.

Values of H_q for sample sizes other than 5 can be found by the relation $H_{q(n\neq 5)} = H_{q(n=5)}\left(\frac{\sigma_T(n\neq 5)}{\sigma_T(n=5)}\right)$ for a fixed $ARL_0 = 200$. From Tables IV–IX, we can conclude that

1. for fixed values of n, K_q, and ARL_0, the value of H_q decreases with an increase in the value of λ_q and vice versa;
2. for fixed values of n, λ_q, and ARL_0, larger values of K_q are giving smaller ARL_1 values for detecting a positive shift in the process dispersion;

Table IV. *ARL* values for the CS-EWMA chart with $\lambda_q = 0.05$, $n = 5$ and $ARL_0 = 200$

τ	$K_q = 0.1$ $H_q = 62.6$	$K_q = 0.25$ $H_q = 47.06$	$K_q = 0.5$ $H_q = 29.5$	$K_q = 0.75$ $H_q = 18.15$	$K_q = 1$ $H_q = 10.62$
0.5	23.156	20.626	17.482	15.133	13.289
0.6	26.652	23.825	20.32	17.749	15.758
0.7	32.317	28.983	24.993	22.144	19.969
0.8	43.072	38.98	34.238	31.092	28.876
0.9	73.716	68.713	63.844	61.61	61.053
0.95	127.175	123.052	120.583	121.6	123.108
1	199.718	200.262	200.951	200.69	199.752
1.05	94.196	88.525	84.496	82.271	81.696
1.1	52.536	47.005	41.056	37.801	35.939
1.2	31.726	27.331	22.219	18.725	16.32
1.3	24.789	21.241	16.942	13.848	11.534
1.4	21.141	18.078	14.341	11.569	9.429
1.5	18.774	16.091	12.706	10.183	8.233
2	13.481	11.562	9.112	7.224	5.712
3	10.16	8.729	6.877	5.442	4.271

Table V. *ARL* values for the CS-EWMA chart with $\lambda_q = 0.1$, $n = 5$, and $ARL_0 = 200$

τ	$K_q = 0.1$ $H_q = 48.1$	$K_q = 0.25$ $H_q = 35.5$	$K_q = 0.5$ $H_q = 22.27$	$K_q = 0.75$ $H_q = 14.1$	$K_q = 1$ $H_q = 8.81$
0.5	17.682	15.366	12.769	10.995	9.717
0.6	20.542	17.895	14.96	13.009	11.626
0.7	25.299	22.109	18.707	16.487	15.072
0.8	34.747	30.731	26.633	24.372	23.188
0.9	63.816	58.764	55.477	55.787	57.655
0.95	115.831	113.49	115.278	120.601	126.425
1	199.304	199.885	200.85	199.751	200.113
1.05	102.296	96.919	95.415	95.943	96.935
1.1	55.637	50.376	45.925	44.641	44.409
1.2	30.526	26.334	22.075	19.62	18.345
1.3	22.596	19.244	15.62	13.331	11.874
1.4	18.601	15.786	12.672	10.624	9.178
1.5	16.218	13.729	10.956	9.064	7.73
2	11.075	9.385	7.424	6.039	4.997
3	8.098	6.881	5.438	4.395	3.559

Table VI. *ARL* values for the CS-EWMA chart with $\lambda_q = 0.2$, $n = 5$, and $ARL_0 = 200$

τ	$K_q = 0.1$ $H_q = 35.2$	$K_q = 0.25$ $H_q = 24.96$	$K_q = 0.5$ $H_q = 15.47$	$K_q = 0.75$ $H_q = 10.03$	$K_q = 1$ $H_q = 6.53$
0.5	14.017	11.652	9.421	8.106	7.222
0.6	16.572	13.807	11.243	9.796	8.858
0.7	20.988	17.597	14.579	12.978	12.141
0.8	30.235	25.828	22.383	21.258	21.411
0.9	59.291	54.54	54.423	59.188	65.493
0.95	112.046	111.302	120.099	132.689	145.693
1	200.201	199.572	200.733	200.695	200.392
1.05	104.94	100.801	100.762	103.131	104.378
1.1	56.775	51.104	48.576	48.77	49.734
1.2	29.533	24.981	21.284	19.8	19.186
1.3	20.761	17.182	13.998	12.367	11.485
1.4	16.545	13.588	10.88	9.367	8.401
1.5	14.048	11.505	9.131	7.731	6.791
2	9.041	7.414	5.805	4.797	4.078
3	6.369	5.252	4.117	3.372	2.797

Table VII. *ARL* values for the CS-EWMA chart with $\lambda_q = 0.3$, $n = 5$, and $ARL_0 = 200$

τ	$K_q = 0.1$ $H_q = 28.13$	$K_q = 0.25$ $H_q = 19.37$	$K_q = 0.5$ $H_q = 11.87$	$K_q = 0.75$ $H_q = 7.78$	$K_q = 1$ $H_q = 5.16$
0.5	12.39	9.956	7.911	6.822	6.134
0.6	14.876	11.99	9.621	8.444	7.772
0.7	19.235	15.661	12.875	11.729	11.362
0.8	28.467	23.896	21.086	21.167	22.79
0.9	57.66	53.647	56.972	66.231	78.395
0.95	111.3	112.994	128.065	147.259	164.091
1	200.262	199.154	199.005	200.568	199.814
1.05	105.433	101.759	102.912	105.905	106.502
1.1	56.456	50.968	49.573	51.114	52.289
1.2	28.726	23.988	20.749	19.841	19.724
1.3	19.793	16.096	13.122	11.908	11.206
1.4	15.467	12.395	9.873	8.664	7.917
1.5	12.948	10.325	8.125	6.969	6.265
2	7.991	6.376	4.933	4.108	3.535
3	5.473	4.409	3.412	2.789	2.366

Table VIII. *ARL* values for the CS-EWMA chart with $\lambda_q = 0.4$, $n = 5$, and $ARL_0 = 200$

τ	$K_q = 0.1$ $H_q = 23.43$	$K_q = 0.25$ $H_q = 15.83$	$K_q = 0.5$ $H_q = 9.62$	$K_q = 0.75$ $H_q = 6.308$	$K_q = 1$ $H_q = 4.24$
0.5	11.425	8.956	7.014	6.056	5.526
0.6	13.878	10.937	8.707	7.703	7.26
0.7	18.23	14.579	12.041	11.259	11.47
0.8	27.332	22.911	20.802	22.072	25.63
0.9	56.545	53.599	61.015	76.068	94.607
0.95	110.015	115.617	138.526	161.045	184.439
1	199.007	200.336	200.485	199.411	199.888
1.05	104.542	102.206	105.385	106.762	107.729
1.1	55.908	50.87	50.536	52.258	53.902
1.2	28.069	23.363	20.537	19.92	20.051
1.3	19.113	15.296	12.611	11.57	11.075
1.4	14.786	11.625	9.3	8.193	7.625
1.5	12.235	9.581	7.494	6.484	5.892
2	7.32	5.716	4.373	3.647	3.158
3	4.884	3.852	2.965	2.414	2.134

Table IX. *ARL* values for the CS-EWMA chart with $\lambda_q = 0.5$, $n = 5$, and $ARL_0 = 200$

τ	$K_q = 0.1$ $H_q = 20.1$	$K_q = 0.25$ $H_q = 13.25$	$K_q = 0.5$ $H_q = 7.99$	$K_q = 0.75$ $H_q = 5.27$	$K_q = 1$ $H_q = 3.57$
0.5	10.819	8.244	6.394	5.563	5.16
0.6	13.25	10.215	8.096	7.292	7.082
0.7	17.575	13.835	11.575	11.222	12.113
0.8	26.704	22.207	21.05	23.96	30.154
0.9	56.118	53.989	65.611	86.907	113.788
0.95	111.12	118.942	146.808	177.365	205.588
1	200.775	199.405	199.3062	199.499	200.476
1.05	104.617	102.52	105.77	107.044	106.577
1.1	55.436	50.59	51.073	53.56	54.65
1.2	27.554	22.734	20.376	20.218	20.334
1.3	18.626	14.656	12.186	11.379	11.054
1.4	14.272	11.049	8.842	7.888	7.449
1.5	11.76	9.015	7.041	6.138	5.654
2	6.863	5.226	3.98	3.31	2.91
3	4.487	3.464	2.601	2.22	1.87

3. for fixed values of n, λ_q, and ARL_0, negative shifts of smaller magnitude are detected efficiently using moderate values of K_q like $0.25 \leqslant K_q \leqslant 0.5$, whereas for the larger shifts in the negative direction, larger values of K_q are recommended;
4. for fixed values of n, K_q, and ARL_0, larger values of λ_q are recommended for detecting larger positive shifts and vice versa, whereas for negative shifts, varied behavior is seen;
5. for fixed values of n, λ_q, and ARL_0, the value of H_q decreases with an increase in the value of K_q and vice versa;

In this article, we have used the statistic T_j to design the control structure of our proposed chart. Many other transformations of S_j^2 that result into a statistic, which is distributed approximately normal, can be used. Acosta-Mejia *et al.*[4] proposed two such transformations named as P_σ and χ. Castagliola *et al.*[13] proposed a four-parameter Johnston transformation and named the resulting variable as U_k. Huwang *et al.*[14] proposed a logarithmic transformation and showed that $Y_j = \ln\left(S_j^2/\sigma_0^2\right)$ follows approximately a normal distribution, which can be used to design a memory control chart to monitor the process dispersion. In the next section, we compare the control charts based on these transformations with our proposal.

4. Comparisons

This section contains the comparison of the proposed CS-EWMA chart with the S^2-EWMA, CUSUM-S^2, and some other recently proposed CUSUM and EWMA charts for monitoring the process dispersion.

CS-EWMA versus S^2-EWMA: The *ARL* values for the S^2-EWMA chart (discussed in Section 2.1) are given in Table II. Comparison reveals that the performance of the proposed chart with $K_q = 1$ is almost the same as compared with the S^2-EWMA for the positive shifts, but for negative shifts, the performance of the CS-EWMA chart is far more superior than that of the S^2-EWMA. Moreover, the performance of S^2-EWMA becomes very poor for moderate and large values of λ, even the ARL_1 values become larger than the prefixed ARL_0 for negative shifts. This is not the case with the proposed CS-EWMA chart, as for larger values of λ_q, the CUSUM factor in the proposed chart still makes it remain better in terms of *ARL*s (cf. Table II vs. Tables IV–IX).

CS-EWMA versus CUSUM-S^2: Table III contains the *ARL* values for the CUSUM-S^2 chart proposed by Castagliola *et al.*[9] This CUSUM-S^2 chart is a special form of our proposed CS-EWMA chart with $\lambda_q = 1$. The performance of the proposed chart is better than that of the CUSUM-S^2 chart, especially for the smaller values of smoothing constant λ_q. The additional parameter (i.e. λ_q) in CS-EWMA chart makes sure that the performance of the proposed chart is not deflated for the larger values of K_q like it occurs with the CUSUM-S^2 chart for larger values of K (cf. Table III vs. Tables IV–IX).

CS-EWMA versus some other EWMA and CUSUM charts: Acosta-Mejia *et al.*[4] proposed some CUSUM-type charts (named as χ CUSUM, P_σ CUSUM, and *CP* CUSUM) for monitoring process dispersion and compared the performance of their proposed charts with the CUSUM *R* chart by Page,[1] CUSUM *S* chart by Tuprah and Ncube[2] and CUSUM ln S^2 chart by Chang and Gan.[3] They showed through comparison that their proposed *CP* CUSUM is performing better (in terms of *ARL* values) than do the other charts discussed. Similarly, Huwang *et al.*[14] proposed two new EWMA-type charts (named as *HHW*1 CUSUM and *HHW*2 CUSUM) for monitoring the standard deviation of a process. They also compared the performance of their proposed chart with some other competitors, like the *CH* EWMA chart by Crowder and Hamilton[7] and the *SJ* EWMA chart by Shu and Jiang,[15] and showed that their proposed charts are more sensitive (in detecting shifts) than the other competitors. The *ARL* values of the charts discussed by Acosta-Mejia *et al.*[4] are given in Table X, whereas Table XI contains the *ARL*s of the charts discussed by Huwang *et al.*[14]

All charts presented in Tables X and XI are one-sided, i.e. designed to detect just positive shift in the process dispersion. For a valid comparison of the proposed chart with these charts, we have evaluated the *ARL* values of the proposed chart with the one-sided structure for monitoring an increase in process standard deviation. These *ARL*s are given in Table XII with $K_q = 1$ and $ARL_0 = 200$.

Table X. *ARL* values for some one-sided CUSUM-type charts for detecting variance increases with $n = 5$ and $ARL_0 = 200$

τ	CUSUM ln S^2 $K = 0.068$ $H = 2.66$	CUSUM R $K = 2.56$ $H = 4.88$	χ CUSUM $K = 0.38$ $H = 4.28$	P_σ CUSUM $K = 0.38$ $H = 4.28$	CUSUM-S $K = 1.1034$ $H = 1.90$	CP CUSUM $K = 1.193$ $H = 18.45$
1	199.93	201.80	200.70	201.10	200.60	200.76
1.1	42.94	40.40	41.04	41.04	38.80	34.60
1.2	18.07	17.60	17.17	17.15	16.85	14.14
1.3	10.75	10.82	10.23	10.21	10.36	8.42
1.4	7.63	7.81	7.26	7.24	7.50	5.93
1.5	5.98	6.13	5.66	5.65	5.85	4.58
2	3.18	3.13	2.90	2.98	3.01	2.20

Table XI. *ARL* values for some one-sided EWMA-type charts for detecting variance increases with $n = 5$ and $ARL_0 = 200$

	$\lambda = 0.05$				$\lambda = 0.1$			
τ	CH-EWMA ($L = 1.055$)	SJ-EWMA ($L = 1.568$)	HHW1-EWMA ($L = 1.828$)	HHW2-EWMA ($L = 1.872$)	CH-EWMA ($L = 1.303$)	SJ-EWMA ($L = 1.943$)	HHW1-EWMA ($L = 2.079$)	HHW2-EWMA ($L = 2.139$)
1	200.33	200.75	200.92	199.57	200.02	200.36	199.51	200.35
1.1	43.24	32.26	28.89	27.28	44.26	35.15	34.32	32.05
1.2	18.09	14.43	11.69	10.78	18.23	14.96	14.1	12.69
1.3	10.77	9.17	6.85	6.2	10.56	9.09	8.2	7.21
1.4	7.63	6.73	4.75	4.24	7.35	6.53	5.65	4.89
1.5	5.98	5.38	3.62	3.22	5.68	5.13	4.28	3.68
2	3.18	2.93	1.8	1.62	2.95	2.72	2.03	1.76
	$\lambda = 0.2$				$\lambda = 0.3$			
	CH-EWMA ($L = 1.513$)	SJ-EWMA ($L = 2.270$)	HHW1-EWMA ($L = 2.253$)	HHW2-EWMA ($L = 2.355$)	CH-EWMA ($L = 1.598$)	SJ-EWMA ($L = 2.433$)	HHW1-EWMA ($L = 2.302$)	HHW2-EWMA ($L = 2.477$)
1	200.64	199.48	199.43	200.65	199.4	199.67	200.22	199.45
1.1	46.63	39.73	41.18	37.87	48.48	43.45	46.14	41.79
1.2	18.79	16.05	16.66	14.7	19.52	17.25	18.65	16.2
1.3	10.54	9.21	9.45	8.16	10.67	9.56	10.35	8.82
1.4	7.16	6.4	6.45	5.49	7.09	6.43	6.9	5.81
1.5	5.41	4.89	4.83	4.07	5.24	4.8	5.11	4.26
2	2.67	2.45	2.24	1.88	2.47	2.3	2.32	1.93

Table XII. *ARL* values for one-sided CS-EWMA chart with $K_q = 1$, $n = 5$, and $ARL_0 = 200$

τ	$\lambda_q = 0.05$ $H_q = 5.39$	$\lambda_q = 0.1$ $H_q = 5.13$	$\lambda_q = 0.2$ $H_q = 4.54$	$\lambda_q = 0.3$ $H_q = 3.928$	$\lambda_q = 0.4$ $H_q = 3.42$	$\lambda_q = 0.5$ $H_q = 3.01$
1	200.4035	200.9845	200.7247	200.1826	200.621	199.6103
1.1	23.963	31.478	36.48	39.44	41.226	43.263
1.2	11.261	13.798	15.39	16.031	16.599	17.147
1.3	8.006	9.1	9.485	9.553	9.635	9.684
1.4	6.53	7.041	7.012	6.837	6.721	6.641
1.5	5.685	5.941	5.689	5.432	5.224	5.112
2	3.975	3.834	3.423	3.08	2.858	2.673
3	3.04	2.732	2.347	2.143	1.894	1.674

It can be noticed through the comparison that the proposed chart is outperforming all its competitors in the form of EWMA-type and CUSUM-type charts (cf. Tables X and XI vs. Table XII).

Finally, before concluding this section, we provide the *ARL* curves of the different charts discussed earlier. Figure 1 contains the *ARL* curves of the two-sided charts containing: CS-EWMA chart with $\lambda_q = 0.2$, $K_q = 0.25$, and $H_q = 24.96$; S^2-EWMA chart with $\lambda = 0.2$ and

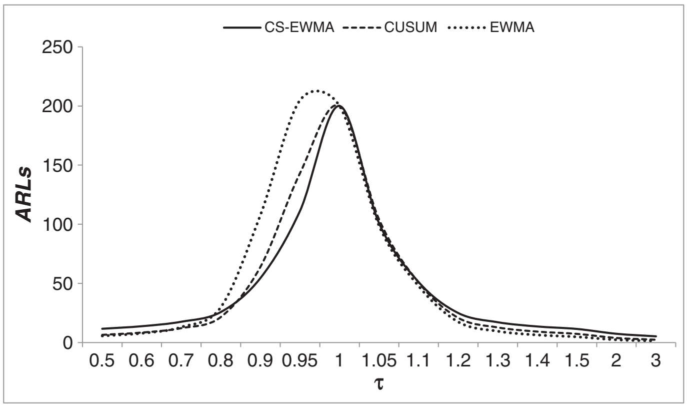

Figure 1. *ARL* curves for two-sided structures of CS-EWMA, S^2-EWMA, and CUSUM-S^2 charts with $ARL_0 = 200$

$L = 2.592$; and CUSUM-S^2 chart with $K = 0.25$ and $H = 6.476$. Figure 2 shows the *ARL* curves of the one-sided charts for positive shifts containing CS-EWMA chart with $\lambda_q = 0.05$, $K_q = 1$, and $H_q = 5.39$; *CP* CUSUM chart with $K = 1.193$ and $H = 18.45$; CUSUM *S* chart with $K = 1.1034$ and $H = 1.90$; *CH* EWMA chart with $\lambda = 0.05$ and $L = 1.055$; *SJ* EWMA chart with $\lambda = 0.05$ and $L = 1.568$; and *HHW*2 EWMA chart with $\lambda = 0.05$ and $L = 1.872$.

Figure 1 shows that the performance of all three charts is almost the same for positive shifts, but for negative shifts, the proposed chart is giving a better *ARL* performance. Similarly, in Figure 2, the *ARL* curve of the proposed chart seems lower than all other curves for small values of τ and thus showing a better performance for small positive shifts.

5. Illustrative example

An application of CS-EWMA chart on a simulated dataset is provided in this section to show the implementation of the proposal. For this purpose, two datasets are generated having 40 samples of size $n = 5$ for both the samples. The first 20 samples are generated from $N(10, 4)$, referring to an in-control situation with $\mu = 10$ and $\sigma_0^2 = 4$. For dataset 1, the remaining 20 observations are generated from $N(10, 5)$, showing a positive shift in the process standard deviation with $\tau = \sqrt{5}/\sqrt{4} = 1.118$. Similarly, for dataset 2, the remaining 20 observations are generated from $N(10, 3)$, showing a negative shift in dispersion with $\tau = 0.866$. Now, the S^2-EWMA chart, CUSUM-S^2 chart, and their mixture named as CS-EWMA chart are applied to the given datasets. The chart output of the S^2-EWMA chart with $\lambda = 0.2$ and $L = 2.592$ is shown in Figure 3. Figure 4 shows the graphical display of the CUSUM-S^2 with $K = 0.5$ and $H = 3.855$. The calculation details for the proposed chart with $\lambda_q = 0.2$, $K_q = 0.5 \Rightarrow K'_q = 0.167$, and $H_q = 15.47 \Rightarrow H'_q = 5.157$ are given in Table XIII, whereas the chart output is given in Figure 5.

Figures 3–5 clearly indicate that all three charts are giving out-of-control signals at samples 39 and 40 for dataset 1 (i.e. positive shift in the process dispersion). Moreover, the proposed CS-EWMA chart detected a negative shift for dataset 2 at samples 37, 38, 39, and 40, but the other two charts did not signal the downward shift in the process standard deviation.

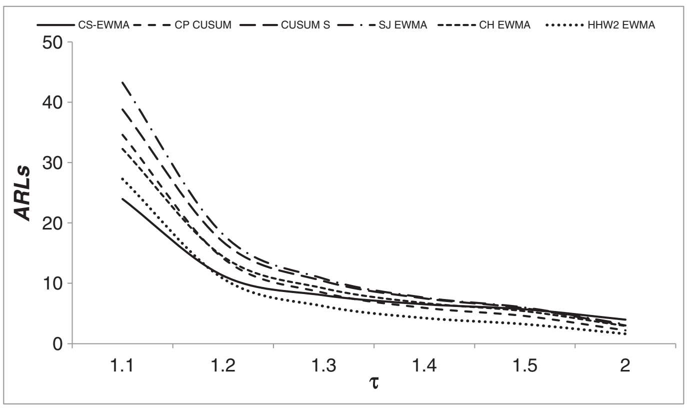

Figure 2. *ARL* curves for one-sided structures of CS-EWMA, *CP* CUSUM, CUSUM-*S*, *SJ* EWMA, *CH* EWMA, and *HHW*2 EWMA charts with $ARL_0 = 200$

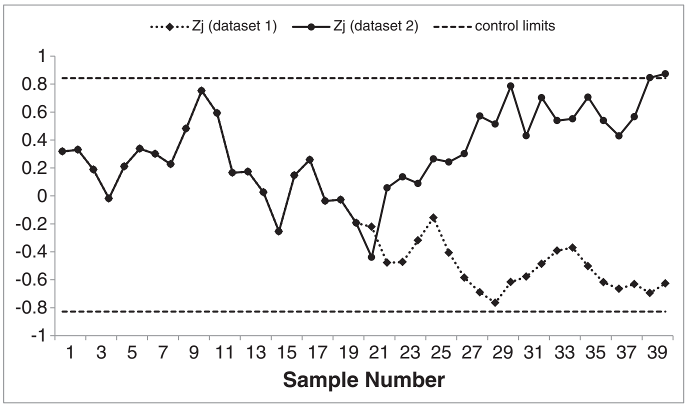

Figure 3. Graphical display of the S^2-EWMA chart with $\lambda = 0.2$ and $L = 2.592$

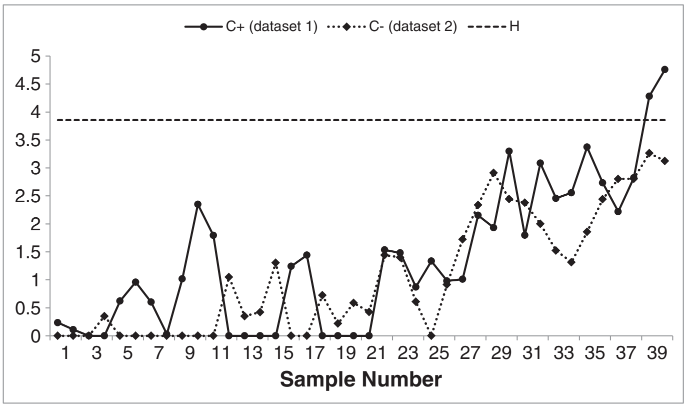

Figure 4. Graphical display of the CUSUM-S^2 chart with K = 0.5 and $H = 3.855$

Table XIII. Calculation details of the proposed CS-EWMA chart for dataset 1

Sample no.	S_j^2	T_j	Q_j	M_j^+	M_j^-	Sample no.	S_j^2	T_j	Q_j	M_j^+	M_j^-
1	5.61	0.74	0.32	0.14	0	21	0.81	−1.42	−0.44	0	0.32
2	4.48	0.38	0.33	0.3	0	22	11.47	2.04	0.06	0	0.1
3	2.58	−0.38	0.19	0.31	0	23	4.69	0.45	0.14	0	0
4	1.7	−0.84	−0.02	0.12	0	24	3.21	−0.1	0.09	0	0
5	7.04	1.13	0.21	0.16	0	25	6.42	0.97	0.26	0.09	0
6	5.96	0.84	0.34	0.32	0	26	3.84	0.15	0.24	0.16	0
7	3.84	0.15	0.3	0.45	0	27	4.94	0.54	0.3	0.29	0
8	3.29	−0.07	0.23	0.5	0	28	9.35	1.65	0.57	0.68	0
9	8.62	1.5	0.48	0.81	0	29	4.21	0.29	0.51	1.02	0
10	10.33	1.84	0.75	1.39	0	30	10.5	1.87	0.79	1.64	0
11	3.33	−0.05	0.59	1.81	0	31	1.45	−0.99	0.43	1.89	0
12	0.65	−1.54	0.17	1.8	0	32	10.1	1.8	0.7	2.42	0
13	3.99	0.21	0.17	1.8	0	33	3.16	−0.12	0.54	2.79	0
14	2.21	−0.57	0.03	1.65	0	34	5.16	0.61	0.55	3.16	0
15	0.88	−1.38	−0.25	1.22	0.1	35	7.84	1.32	0.71	3.7	0
16	9.86	1.75	0.15	1.19	0	36	3.14	−0.13	0.54	4.06	0
17	5.48	0.71	0.26	1.28	0	37	3.44	−0.01	0.43	4.32	0
18	1.1	−1.22	−0.04	1.07	0	38	6.96	1.11	0.57	4.71	0
19	3.48	0.01	−0.03	0.86	0	39	11.04	1.97	0.85	5.38*	0
20	1.67	−0.86	−0.19	0.5	0.03	40	6.47	0.98	0.87	6.08*	0

*An out-of-control signal by CS-EWMA chart

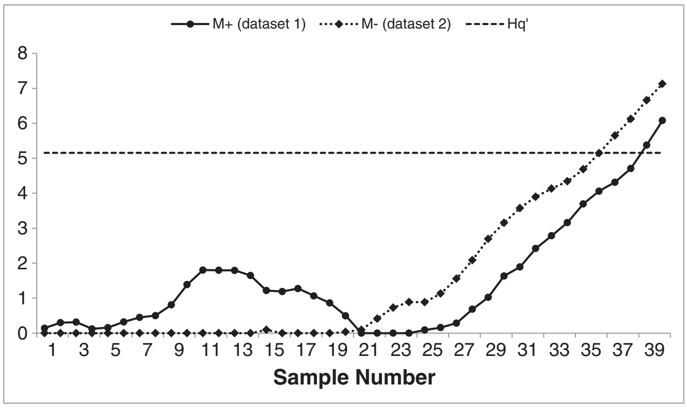

Figure 5. Graphical display of the CS-EWMA chart with $\lambda_q = 0.2$, $K_q = 0.5$ and $H_q = 15.47$

6. Summary and conclusions

The control chart is one of the major tools of statistical process control that is used to differentiate between common and special cause variations. CUSUM and EWMA control charts (also classified as memory control charts) are two popular choices for detecting small and moderate shifts in the process parameters. This article proposes a new memory-type control chart, named as the CS-EWMA chart, for monitoring the process standard deviation. The design of the proposed chart is based on mixing the effects of EWMA and CUSUM control charts into a single control structure. The performance of the proposed CS-EWMA chart is evaluated using *ARL* as indicator. In terms of *ARL* values, the proposed chart is compared with existing CUSUM and EWMA control charts, and it is noticed that the proposed chart has better performance for both positive as well as negative shifts in the process dispersion. Finally, an illustrative example is provided to show the application of the proposed chart.

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