Robust Location Estimators for the $\overline{X}$ Control Chart

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This article studies estimation methods for the location parameter. We consider several robust location estimators as well as several estimation methods based on a phase I analysis, i.e., the use of a control chart to study a historical dataset retrospectively to identify disturbances. In addition, we propose a new type of phase I analysis. The estimation methods are evaluated in terms of their mean-squared errors and their effect on the $\overline{X}$ control charts used for real-time process monitoring (phase II). It turns out that the phase I control chart based on the trimmed trimean far outperforms the existing estimation methods. This method has therefore proven to be very suitable for determining $\overline{X}$ phase II control chart limits.

Key Words: Estimation; Mean; Phase I; Phase II; Robust; Shewhart Control Chart; Statistical Process Control; Trimean.

Introduction

THE PERFORMANCE of a process depends on the stability of its location and dispersion parame-

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ters, and an optimal performance requires that any change in these parameters should be detected as soon as possible. To monitor a process with respect to these parameters, Shewhart introduced the idea of control charts in the 1920s. The present paper focuses on phase I location-estimation methods for constructing the location control chart.

Let Y_{ij}, $i = 1, 2, 3, \ldots$ and $j = 1, 2, \ldots, n$, denote phase II samples of size n taken in sequence of the process variable to be monitored. We assume the Y_{ij}'s to be independent and $N(\mu + \delta\sigma, \sigma^2)$ distributed, where δ is a constant. When $\delta = 0$, the mean of the process is in control; otherwise, the process mean has changed. Let $\overline{Y}_i = (1/n) \sum_{j=1}^{n} Y_{ij}$ be an estimate of $\mu + \delta\sigma$ based on the ith sample Y_{ij}, $j = 1, 2, \ldots, n$. When the in-control μ and σ are known, the process mean can be monitored by plotting $\overline{Y}_i$ on a control chart with respective upper and

lower control limits

$$\text{UCL} = \mu + C_n \sigma / \sqrt{n},$$
$$\text{LCL} = \mu - C_n \sigma / \sqrt{n}, \qquad (1)$$

where C_n is the factor such that, for a chosen type I error probability p, we have

$$P(\text{LCL} \leq \bar{Y}_i \leq \text{UCL}) = 1 - p.$$

When $\bar{Y}_i$ falls within the control limits, the process is deemed to be in control. We define E_i as the event that $\bar{Y}_i$ falls beyond the limits, $P(E_i)$ as the probability that sample i falls beyond the limits and RL as the run length, i.e., the number of samples until the first $\bar{Y}_i$ falls beyond the limits. When μ and σ are known, the events E_i are independent, and therefore RL is geometrically distributed with parameter $p = P(E_i)$. It follows that the average run length (ARL) is given by $1/p$ and that the standard deviation of the run length (SDRL) is given by $\sqrt{1-p}/p$.

In practice, the process parameters μ and σ are usually unknown. Therefore, they must be estimated from samples taken when the process is assumed to be largely in control. This stage in the control charting process is denoted as phase I (cf., Woodall and Montgomery (1999), Vining (2009)). The resulting estimates determine the control limits that are used to monitor the location of the process in phase II. Define $\hat{\mu}$ and $\hat{\sigma}$ as unbiased estimates of μ and σ, respectively, based on k phase I samples of size n, which are denoted by X_{ij}, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$. The control limits can be estimated by

$$\widehat{\text{UCL}} = \hat{\mu} + C_n \hat{\sigma} / \sqrt{n},$$
$$\widehat{\text{LCL}} = \hat{\mu} - C_n \hat{\sigma} / \sqrt{n}. \qquad (2)$$

Let F_i denote the event that $\bar{Y}_i$ is above $\widehat{\text{UCL}}$ or below $\widehat{\text{LCL}}$. We define $P(F_i \mid \hat{\mu}, \hat{\sigma})$ as the probability that sample i generates a signal given $\hat{\mu}$ and $\hat{\sigma}$, i.e.,

$$P(F_i \mid \hat{\mu}, \hat{\sigma}) = P(\bar{Y}_i < \widehat{\text{LCL}} \text{ or } \bar{Y}_i > \widehat{\text{UCL}} \mid \hat{\mu}, \hat{\sigma}). \qquad (3)$$

Given $\hat{\mu}$ and $\hat{\sigma}$, the distribution of the run length is geometric with parameter $P(F_i \mid \hat{\mu}, \hat{\sigma})$. Consequently, the conditional ARL is given by

$$E(\text{RL} \mid \hat{\mu}, \hat{\sigma}) = \frac{1}{P(F_i \mid \hat{\mu}, \hat{\sigma})}. \qquad (4)$$

In contrast with the conditional RL distribution, the unconditional RL distribution takes into account the random variability introduced into the charting procedure through parameter estimation. It can be obtained by averaging the conditional RL distribution over all possible values of the parameter estimates. The unconditional p is

$$p = E(P(F_i \mid \hat{\mu}, \hat{\sigma})), \qquad (5)$$

and the unconditional average run length is

$$\text{ARL} = E\left(\frac{1}{P(F_i \mid \hat{\mu}, \hat{\sigma})}\right). \qquad (6)$$

Quesenberry (1993) showed, for the $\overline{X}$ control chart, that the unconditional in-control and out-of-control ARL values are higher than in the case where the process parameters are known. Furthermore, a higher in-control ARL is not necessarily better because the RL distribution will reflect an increased number of short RLs as well as an increased number of long RLs. He concluded that, if limits are to behave like known limits, the number of samples in phase I should be at least $400/(n-1)$.

Jensen et al. (2006) conducted a literature survey of the effects of parameter estimation on control-chart properties and identified some issues for future research. One of their main recommendations is to study robust or alternative estimators for μ and σ (e.g., Rocke (1989, 1992), Tatum (1997), Vargas (2003), Davis and Adams (2005)). The effect of using these robust estimators on phase II should also be assessed (Jensen et al. (2006, p. 360)). These recommendations are the subject of the present paper, i.e., we will examine alternative location-estimation methods as well as the impact of these estimators on the phase II performance of the $\overline{X}$ control chart.

So far the literature has proposed several alternative robust location estimators. Rocke (1989) proposed the 25% trimmed mean of the sample means, the median of the sample means, and the mean of the sample medians. Rocke (1992) gave the practical details for the construction of the charts based on these estimators. Alloway and Raghavachari (1991) constructed a control chart based on the Hodges–Lehmann estimator. Tukey (1997) and Wang et al. (2007) developed the trimean estimator, which is defined as the weighted average of the median and the two other quartiles. Finally, Jones-Farmer et al. (2009) proposed a rank-based phase I control chart. Based on this phase I control chart, they define the in-control state of a process and identify an in-control reference sample. The resultant reference sample can be used to estimate the location parameter.

In this article, we compare existing and new methods for estimating the in-control μ. The collection of

methods includes robust sample statistics for location and estimation methods based on phase I control charts (cf., Jones-Farmer et al. (2009)). The methods are evaluated in terms of their mean-squared errors (MSE) and their effect on the $\overline{X}$ phase II control-chart performance. We consider situations where the phase I data are uncontaminated and normally distributed, as well as various types of contaminated phase I situations.

The remainder of the paper is structured as follows. First, we present several phase I sample statistics for the process location and assess the MSE of the estimators. Then we describe some existing phase I control charts and present a new algorithm for phase I analysis. Following that, we present the design schemes for the $\overline{X}$ phase II control chart and derive the control limits. Next, we describe the simulation procedure and present the effect of the proposed methods on the phase II performance. The final section offers some recommendations.

Proposed Location Estimators

To understand the behavior of the estimators it is useful to distinguish two groups of disturbances, namely, diffuse and localized (cf., Tatum (1997)). Diffuse disturbances are outliers that are spread over all of the samples whereas localized disturbances affect all observations in one sample. We include various types of estimators (both robust estimators and several estimation methods based on the principle of control charting) and compare them under various types of disturbances. The first subsection introduces the estimators, while the second subsection presents the MSE of the estimators.

Location Estimators

Recall that X_{ij}, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$, denote the phase I data. The X_{ij}'s are assumed to be independent and largely $N(\mu, \sigma^2)$ distributed. We denote by $X_{i,(v)}$, $v = 1, 2, \ldots, n$, the vth order statistic in sample i.

The first estimator that we consider is the mean of the sample means,

$$\overline{\overline{X}} = \frac{1}{k}\sum_{i=1}^{k}\overline{X}_i = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{1}{n}\sum_{j=1}^{n}X_{ij}\right). \qquad (7)$$

This estimator is included to provide a basis for comparison, as it is the most efficient estimator for normally distributed data. However, it is well known that this estimator is not robust against outliers.

We also consider three robust estimators proposed earlier by Rocke (1989): the median of the sample means,

$$M(\overline{X}) = \text{median}(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_k); \qquad (8)$$

the mean of the sample medians,

$$\overline{M} = \frac{1}{k}\sum_{i=1}^{k}M_i, \qquad (9)$$

with M_i the median of sample i; and the trimmed mean of the sample means,

$$\overline{\overline{X}}_\alpha = \frac{1}{k - 2\lceil k\alpha \rceil} \times \left[\sum_{v=\lceil k\alpha \rceil+1}^{k-\lceil k\alpha \rceil}\overline{X}_{(v)}\right], \qquad (10)$$

where α denotes the percentage of samples to be trimmed, $\lceil z \rceil$ denotes the ceiling function, i.e., the smallest integer not less than z, and $\overline{X}_{(v)}$ denotes the vth ordered value of the sample means. In our study, we consider the 20% trimmed mean, which trims the six smallest and the six largest sample means when $k = 30$. Of course, other trimming percentages could have been used. In fact, we have also used 10% and 25%, but the results with 20% are representative for this estimator.

Furthermore, our analysis includes the Hodges–Lehmann estimator (Hodges and Lehmann (1963)), an estimator based on the so-called Walsh averages. The h $(= n(n+1)/2)$ Walsh averages of sample i are

$$W_{i,k,l} = (X_{i,k} + X_{i,l})/2,$$

for $k = 1, 2, \ldots, n$, $l = 1, 2, \ldots, n$, and $k \leq l$. The Hodges–Lehmann estimate for sample i, denoted by HL_i, is defined as the median of the Walsh averages. Alloway and Raghavachari (1991) conducted a Monte Carlo simulation to determine whether the mean or the median of the sample Hodges–Lehmann estimates should be used to determine the final location estimate. They concluded that the mean of the sample values should be used,

$$\overline{\text{HL}} = \frac{1}{k}\sum_{i=1}^{k}\text{HL}_i, \qquad (11)$$

and that the resulting estimate is unbiased.

In this study, we also include the trimean statistic. The trimean of sample i is the weighted average of the sample median and the two other quartiles,

$$\text{TM}_i = (Q_{i,1} + 2Q_{i,2} + Q_{i,3})/4,$$

where $Q_{i,q}$ is the qth quartile of sample i, $q = 1, 2, 3$ (cf., Tukey (1997), Wang et al. (2007)). It also

equals the average of the median and the midhinge $(1/2)[Q_{i,2} + (Q_{i,1} + Q_{i,3})/2]$ (cf., Weisberg (1992)). We use the following definitions for the quartiles: $Q_{i,1} = X_{i,(a)}$ and $Q_{i,3} = X_{i,(b)}$ with $a = \lceil n/4 \rceil$ and $b = n - a + 1$. This means that $Q_{i,1}$ and $Q_{i,3}$ are defined as the second smallest and the second largest observations, respectively, for $4 \leq n \leq 7$, and as the third smallest and the third largest values, respectively, for $8 \leq n \leq 11$. Like the median and the midhinge, but unlike the sample mean, the trimean is a statistically resistant L-estimator (a linear combination of order statistics), with a breakdown point of 25% (see Wang et al. (2007)). According to Tukey (1977), using the trimean instead of the median gives a more useful assessment of location or centering. According to Weisberg (1992), the "statistical resistance" benefit of the trimean as a measure of the center of a distribution is that it combines the median's emphasis on center values with the midhinge's attention to the extremes. The trimean is almost as resistant to extreme scores as the median and is less subject to sampling fluctuations than the arithmetic mean in extremely skewed distributions. Asymptotic distributional results of the trimean can be found in Wang et al. (2007). The location estimate analyzed below is the mean of the sample trimeans, i.e.,

$$\overline{\text{TM}} = \frac{1}{k} \sum_{i=1}^{k} \text{TM}_i. \qquad (12)$$

Finally, we consider a statistic that is expected to be robust against both diffuse and localized disturbances, namely, the trimmed mean of the sample trimeans, defined by

$$\overline{\text{TM}}_\alpha = \frac{1}{k - 2\lceil k\alpha \rceil} \times \left[\sum_{v=\lceil k\alpha \rceil+1}^{k-\lceil k\alpha \rceil} \text{TM}_{(v)} \right], \qquad (13)$$

where $\text{TM}_{(v)}$ denotes the vth ordered value of the

TABLE 1. Proposed Location Estimators

Estimator	Notation
Mean of sample means	$\overline{\overline{X}}$
Median of sample means	$M(\overline{X})$
Mean of sample medians	$\overline{M}$
20% trimmed mean of sample means	$\overline{\overline{X}}_{20}$
Mean of sample Hodges-Lehmann	$\overline{\text{HL}}$
Mean of sample trimeans	$\overline{\text{TM}}$
20% trimmed mean of sample trimeans	$\overline{\text{TM}}_{20}$

sample trimeans. We consider the 20% trimmed trimean, which trims the six smallest and the six largest sample trimeans when $k = 30$.

The estimators outlined above are summarized in Table 1.

Efficiency of the Proposed Estimators

The efficiency of control-charting procedures is often evaluated by comparing the variance of the respective location estimators. We use a procedure similar to what was adopted by Tatum (1997) and consider the MSE of the estimators. The MSE will be estimated as

$$\text{MSE} = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{\hat{\mu}^i - \mu}{\sigma} \right)^2,$$

where $\hat{\mu}^i$ is the value of the unbiased estimate in the ith simulation run and N is the number of simulation runs. We include the uncontaminated case, i.e., the situation where all X_{ij} are from the $N(0,1)$ distribution, as well as five types of disturbances (cf., Tatum (1997)):

1. A model for diffuse symmetric variance disturbances in which each observation has a 95% probability of being drawn from the $N(0,1)$ distribution and a 5% probability of being drawn from the $N(0,a)$ distribution, with $a = 1.5, 2.0, \ldots, 5.5, 6.0$.

2. A model for diffuse asymmetric variance disturbances in which each observation is drawn from the $N(0,1)$ distribution and has a 5% probability of having a multiple of a χ_1^2 variable added to it, with the multiplier equal to $0.5, 1.0, \ldots, 4.5, 5.0$.

3. A model for localized variance disturbances in which observations in 3 out of 30 samples are drawn from the $N(0,a)$ distribution, with $a = 1.5, 2.0, \ldots, 5.5, 6.0$.

4. A model for diffuse mean disturbances in which each observation has a 95% probability of being drawn from the $N(0,1)$ distribution and a 5% probability of being drawn from the $N(b,1)$ distribution, with $b = 0.5, 1.0, \ldots, 9.0, 9.5$.

5. A model for localized mean disturbances in which observations in 3 out of 30 samples are drawn from the $N(a,1)$ distribution, with $a = 0.5, 1.0, \ldots, 5.5, 6.0$.

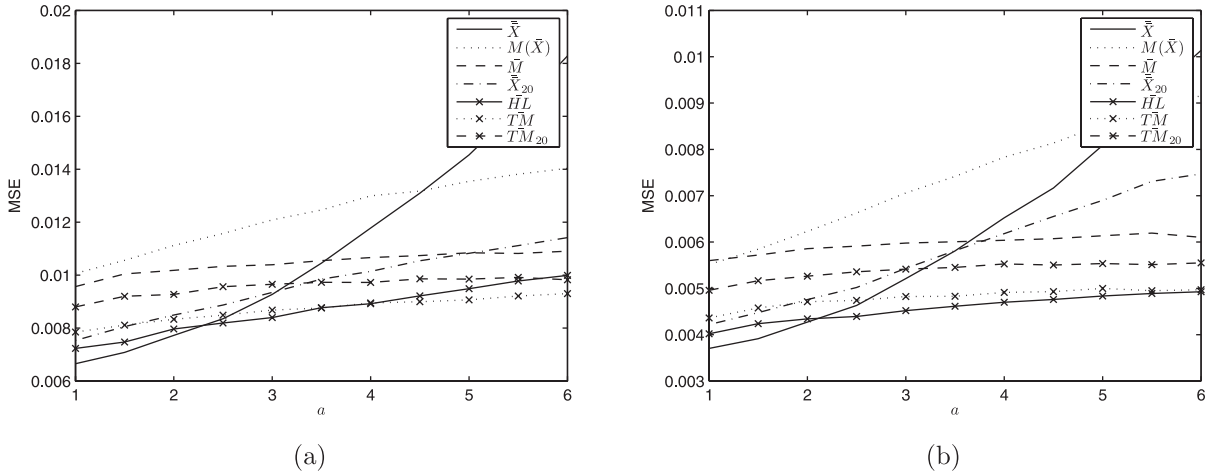The y-axis intercept in Figures 1–5 represents the

FIGURE 1. MSE of Estimators when Symmetric Diffuse Variance Disturbances Are Present for $k = 30$. (a) $n = 5$, (b) $n = 9$.

situation where no contaminations are present. The figures show that, in this situation, the most efficient estimator is $\overline{\overline{X}}$, as was to be expected. The estimators $\overline{HL}$, $\overline{\overline{X}}_{20}$, and $\overline{TM}$ are slightly less efficient, followed by $\overline{TM}_{20}$, $\overline{M}$, and $M(\overline{X})$, the reason being that they use less information.

When diffuse symmetric variance disturbances are present (Figure 1), the best performing estimators are $\overline{HL}$ and $\overline{TM}$. The reason why $\overline{TM}$ performs well in this situation is that it filters out the extreme high and low values in each sample. $\overline{HL}$ also performs well because it obtains the sample statistic using the median of the Walsh averages, which is not sensitive

to outliers. $\overline{M}$ and $\overline{TM}_{20}$ are as efficient in the contaminated situation as in the uncontaminated situation, but they are outperformed by $\overline{HL}$ and $\overline{TM}$ because the latter estimators use more information. It is worth noting that the traditional estimator $\overline{\overline{X}}$ shows relatively bad results despite the symmetric character of the outliers. $M(\overline{X})$ and $\overline{\overline{X}}_{20}$ do not perform very well because these estimators focus on extreme samples whereas, in the present situation, the outliers are spread over all of the samples so that the nontrimmed samples are also infected.

When asymmetric variance disturbances are present (Figure 2), the most efficient estimators are $\overline{TM}$,
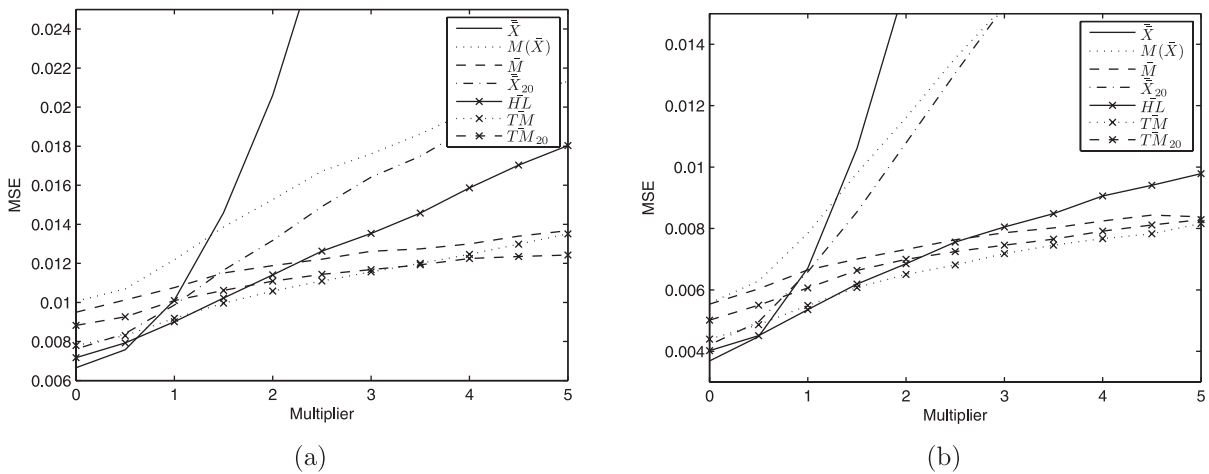

FIGURE 2. MSE of Estimators when Asymmetric Diffuse Variance Disturbances Are Present for $k = 30$. (a) $n = 5$, (b) $n = 9$.
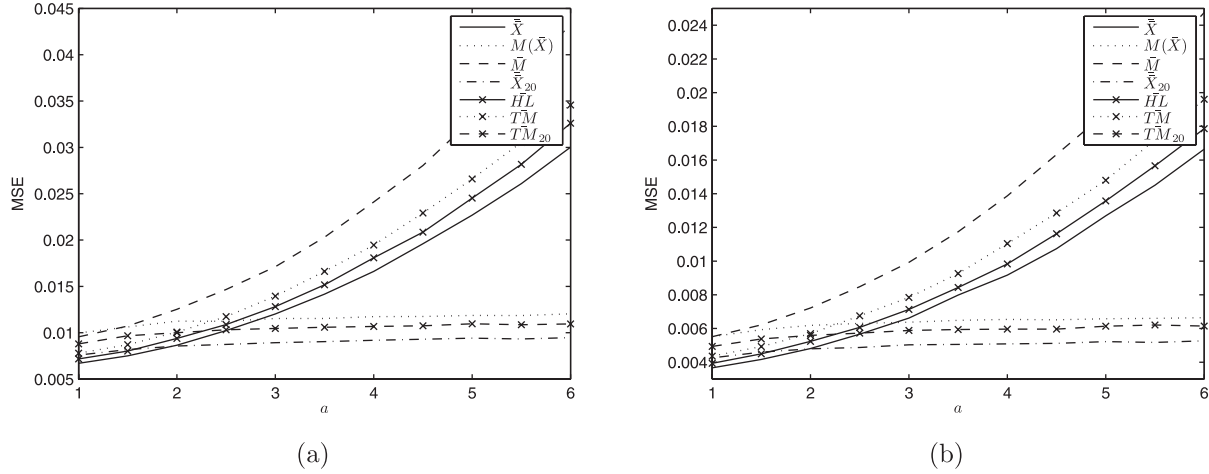
FIGURE 3. MSE of Estimators when Localized Variance Disturbances Are Present for $k = 30$. (a) $n = 5$, (b) $n = 9$.

$\overline{\text{TM}}_{20}$, $\overline{\text{HL}}$, and $\bar{M}$, performing particularly well relative to the other estimators for larger sample sizes. As for the symmetric diffuse case, the estimators that include a method to trim observations within a sample perform better than the methods that focus on sample trimming.

In the case of localized variance disturbances (Figure 3), the estimators based on the principle of trimming sample means rather than within-sample observations—$\overline{\overline{X}}_{20}$, $\overline{\text{TM}}_{20}$, and $M(\bar{X})$—have the lowest MSE . The estimators $\overline{\overline{X}}$, $\overline{\text{HL}}$, $\overline{\text{TM}}$ and, in particular, $\bar{M}$ are less successful because these statistics only perform well if no more than a few observations in a sample are infected rather than all observations, as is the case here.

When diffuse mean disturbances are present (Figure 4), the results are comparable with the situation where there are diffuse asymmetric variance disturbances: $\bar{M}$, $\overline{\text{TM}}_{20}$, and $\overline{\text{TM}}$ perform best, followed by $\overline{\text{HL}}$. Note that, in this situation, $\overline{\overline{X}}$, $M(\bar{X})$, and $\overline{\overline{X}}_{20}$ perform badly.

When localized mean disturbances are present (Figure 5), the results are comparable with the situation where there are localized variance disturbances: the estimators based on the principle of trimming sample means, namely $\overline{\overline{X}}_{20}$, $M(\bar{X})$, and $\overline{\text{TM}}_{20}$, perform best.

To summarize, $\bar{M}$, $\overline{\text{TM}}$, and $\overline{\text{TM}}_{20}$ have the lowest MSE when there are diffuse disturbances. $\bar{M}$ and $\overline{\text{TM}}$
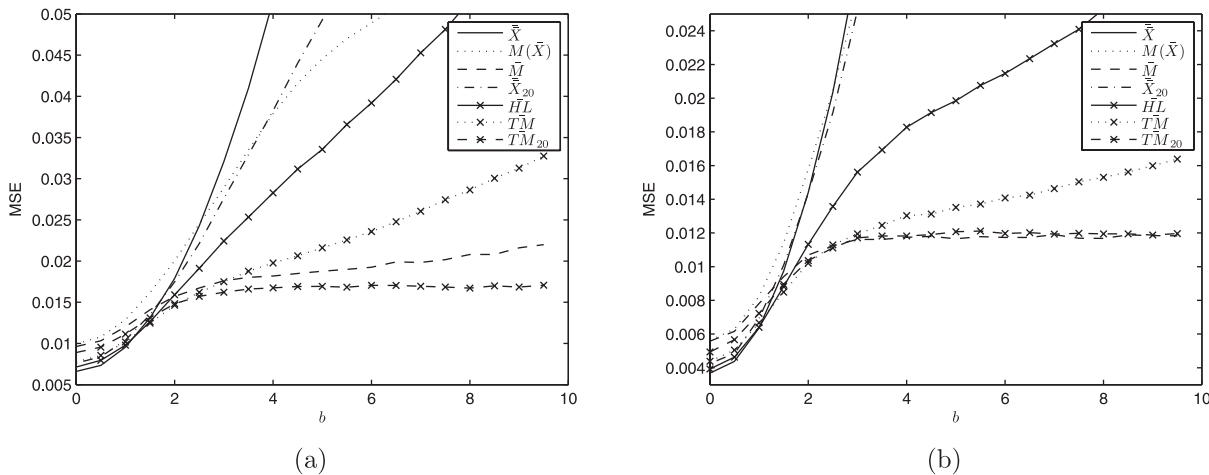

FIGURE 4. MSE of Estimators when Diffuse Mean Disturbances Are Present for $k = 30$. (a) $n = 5$, (b) $n = 9$.
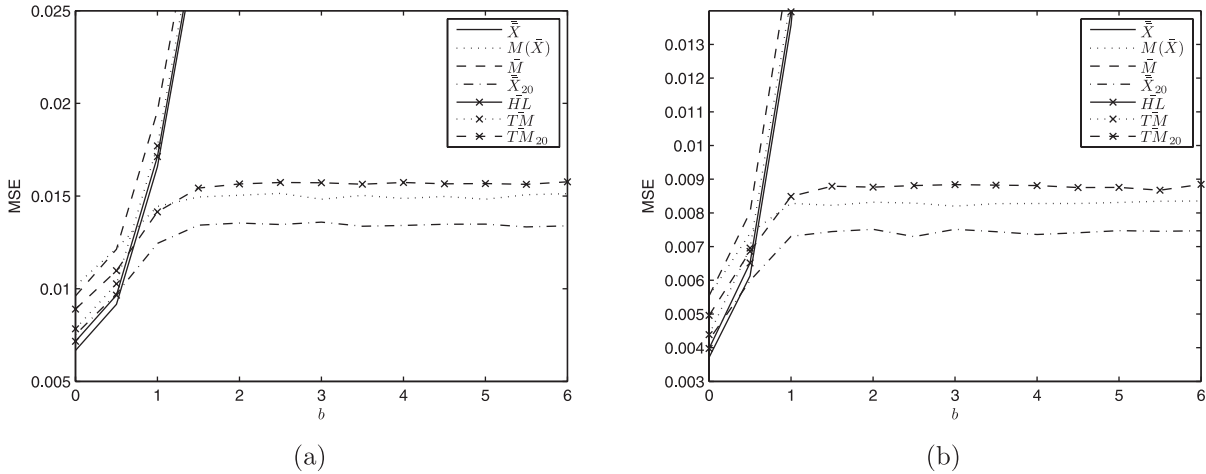
FIGURE 5. MSE of Estimators when Localized Mean Disturbances Are Present for $k = 30$. (a) $n = 5$, (b) $n = 9$.

lose their efficiency advantage when contaminations take the form of localized mean or variance disturbances. In such situations, $M(\bar{X})$, $\overline{\bar{X}}_{20}$, and $\overline{\text{TM}}_{20}$, which involve trimming the sample means, perform relatively well. It is worth noting that $\overline{\text{TM}}_{20}$ has the best performance overall because it is reasonably robust against all types of contaminations.

Proposed Phase I Control Chart Location Estimators

In-control process parameters can be obtained not only via robust statistics but also via phase I control charting. In the latter case, control charts are used retrospectively to study a historical dataset and determine samples that are deemed out of control. Estimates of the process parameters are then based on the in-control samples. In this section, we consider several phase I analyses based on the principle of control charting in order to generate robust estimates of process location. We study the phase I control chart based on the commonly used estimator $\overline{\bar{X}}$ and a phase I control chart based on the mean rank proposed by Jones-Farmer et al. (2009). Moreover, we propose two new types of phase I analyses. The next section presents the various phase I control charts and the following section shows the MSE of the proposed estimation methods.

Phase I Control Charts

The standard procedure in practice is to use the estimator $\overline{\bar{X}}$ for constructing the $\bar{X}$ phase I control-chart limits. The respective upper and lower control limits of the phase I chart are given by $\widehat{\text{UCL}}_{\overline{\bar{X}}} =$

$\overline{\bar{X}} + 3\hat{\sigma}/\sqrt{n}$ and $\widehat{\text{LCL}}_{\overline{\bar{X}}} = \overline{\bar{X}} - 3\hat{\sigma}/\sqrt{n}$, where we estimate σ by the robust standard-deviation estimator proposed by Tatum (1997), using the corrected normalizing constants presented in Schoonhoven et al. (2011). The samples for which $\bar{X}_i$ falls above $\widehat{\text{UCL}}_{\bar{X}}$ or below $\widehat{\text{LCL}}_{\bar{X}}$ are eliminated from the phase I dataset. The final location estimate is the mean of the sample means of the remaining samples:

$$\overline{\bar{X}}' = \frac{1}{k'} \sum_{i \epsilon K} \bar{X}_i \times I_{\widehat{\text{LCL}}_{\overline{\bar{X}}} \leq \bar{X}_i \leq \widehat{\text{UCL}}_{\overline{\bar{X}}}}(\bar{X}_i), \qquad (14)$$

with K the set of samples that are not excluded, k' the number of nonexcluded samples, and I the indicator function. In the following, this adaptive trimmed mean estimator is denoted by $\text{ATM}_{\overline{\bar{X}}}$.

We also consider a phase I analysis that is based on the mean rank proposed by Jones-Farmer et al. (2009). It is a nonparametric estimation method, in which the observations from the k mutually exclusive samples of size n are treated as a single sample of $N = n \times k$ observations. Let $R_{ij} = 1, 2, \ldots, N$ denote the integer rank of observation X_{ij} in the pooled sample of size N. Let $\bar{R}_i = (\sum_{j=1}^{n} R_{ij})/n$ be the mean of the ranks in sample i. If the process is in control, the ranks should be distributed evenly throughout the samples. For an in-control process, the mean and variance of $\bar{R}_i$ are

$$E(\bar{R}_i) = \frac{N+1}{2}$$

and

$$\text{Var}(\bar{R}_i) = \frac{(N-n)(N+1)}{12n}.$$

According to the central limit theorem, the random variable

$$Z_i = \frac{\bar{R}_i - E(\bar{R}_i)}{\sqrt{\text{var}(\bar{R}_i)}}$$

follows approximately a standard normal distribution for large values of n. A control chart for these Z_i's can be constructed with center line equal to 0, upper control limit 3, and lower control limit -3. The samples with Z_i outside the phase I control limits are considered to be out of control and are excluded from the dataset. The location estimate is based on the mean of the remaining sample means,

$$\overline{\overline{X}}^* = \frac{1}{k^*} \sum_{i \epsilon K^*} \bar{X}_i \times I_{-3 \leq Z_i \leq 3}(Z_i), \qquad (15)$$

with K^* the set of samples that are not excluded and k^* the number of nonexcluded samples. This estimation method is denoted by ATM_{MR}.

We now present two new phase I analyses based on the principle of control charting. For the first method, we build a phase I control chart using a robust estimator. The advantage of a robust estimator over a sensitive estimator like $\overline{\overline{X}}$ is that the phase I control limits are not affected by any disturbances so that the correct out-of-control observations are filtered out in phase I. An estimator shown to be very robust by the MSE study in the previous section is $\overline{\text{TM}}_{20}$. A disadvantage is that the estimator is not very efficient under normality. To address this, we use $\overline{\text{TM}}_{20}$ to construct the phase I limits with which we screen $\bar{X}_i$ for disturbances, but then use the efficient estimator $\overline{\overline{X}}$ to obtain the location estimate from the remaining samples. The phase I control limits are given by $\widehat{\text{UCL}}_{\overline{\text{TM}}_{20}} = \overline{\text{TM}}_{20} + 3\hat{\sigma}/\sqrt{n}$ and $\widehat{\text{LCL}}_{\overline{\text{TM}}_{20}} = \overline{\text{TM}}_{20} - 3\hat{\sigma}/\sqrt{n}$, where we estimate σ by Tatum's estimator. We then plot the $\bar{X}_i$'s on the phase I control chart. The samples for which $\bar{X}_i$ falls outside the limits are regarded as out of control and removed from the dataset. The remaining samples are used to determine the grand sample mean,

$$\overline{\overline{X}}^{\#} = \frac{1}{k^{\#}} \sum_{i \epsilon K^{\#}} \bar{X}_i \times I_{\widehat{\text{LCL}}_{\overline{\text{TM}}_{20}} \leq \bar{X}_i \leq \widehat{\text{UCL}}_{\overline{\text{TM}}_{20}}}(\bar{X}_i),$$
$$(16)$$

with $K^{\#}$ the set of samples that are not excluded and $k^{\#}$ the number of nonexcluded samples. The resulting estimator is denoted by $\text{ATM}_{\overline{\text{TM}}_{20}}$.

The fourth type of phase I control chart resembles the chart presented above but employs a different method to screen for disturbances. The procedure consists of two steps.

In the first step, we construct the control chart with limits as we did just before. Note that, for the sake of practical applicability, we use the same factor, namely 3, to derive the $\overline{X}$ and TM charts. We then plot the TM_i's of the phase I samples on the control chart. Charting the TM_i's instead of the $\bar{X}_i$'s ensures that localized disturbances are identified and samples that contain only one single outlier are retained. A location estimate that is expected to be robust against localized mean disturbances is the mean of the sample trimeans of the samples that fall between the control limits

$$\overline{\text{TM}}' = \frac{1}{k^\wedge} \sum_{i \epsilon K^\wedge} \text{TM}_i \times I_{\widehat{\text{LCL}}_{\overline{\text{TM}}_{20}} \leq \text{TM}_i \leq \widehat{\text{UCL}}_{\overline{\text{TM}}_{20}}}(\text{TM}_i),$$

with $K^\wedge$ the set of samples that are not excluded and $k^\wedge$ the number of nonexcluded samples.

Although the remaining phase I samples are expected to be free from localized mean disturbances, they could still contain diffuse disturbances. To eliminate such disturbances, the second step is to screen the individual observations using a phase I individual control chart with respective upper and lower control limits given by $\widehat{\text{UCL}}_{\overline{\text{TM}}'} = \overline{\text{TM}}' + 3\hat{\sigma}$ and $\widehat{\text{LCL}}_{\overline{\text{TM}}'} = \overline{\text{TM}}' - 3\hat{\sigma}$, where σ is estimated by Tatum's estimator. The observations X_{ij} that fall above $\widehat{\text{UCL}}_{\overline{\text{TM}}'}$ or below $\widehat{\text{LCL}}_{\overline{\text{TM}}'}$ are considered out of control and removed from the phase I dataset. The final estimate is the mean of the sample means and is calculated from the observations deemed to be in control,

$$\overline{\overline{X}}'' = \frac{1}{k''} \sum_{i \epsilon K''} \frac{1}{n'_i} \sum_{j \epsilon N'_i} X_{ij} \times I_{\widehat{\text{LCL}}_{\overline{\text{TM}}'} \leq X_{ij} \leq \widehat{\text{UCL}}_{\overline{\text{TM}}'}}(X_{ij}),$$
$$(17)$$

with K'' the set of samples that are not excluded, k'' the number of nonexcluded samples, N'_i the set of observations that are not excluded in sample i, and n'_i the number of nonexcluded observations in sample i. Note that we could also have used the double sum, divided by the sum of the n'_i. The advantage of the applied procedure is that, when a sample is infected by a localized disturbance, the disturbance will have a lower impact on the final location estimate when it is not detected. This estimation method is denoted by $\text{ATM}_{\overline{\text{TM}}'}$.

TABLE 2. Proposed Phase I Analyses

Phase I analyses	Notation
$\overline{\overline{X}}$ control chart with screening	$\text{ATM}_{\overline{\overline{X}}}$
Mean rank control chart with screening	ATM_{MR}
$\overline{\text{TM}}_{20}$ control chart with screening	$\text{ATM}_{\overline{\text{TM}}_{20}}$
$\overline{\text{TM}}'$ control chart with screening	$\text{ATM}_{\overline{\text{TM}}'}$

The proposed phase I analyses are summarized in Table 2.

Efficiency of the Proposed Phase I Control Charts

To determine the efficiency of the proposed phase I control charts, we consider the five types of contaminations defined in our MSE study of the statistics presented in the previous section. The MSE results for the phase I control charts are given in Figures 6–10. To facilitate comparison, we have also included the MSE of the estimators $\overline{\overline{X}}$ and $\overline{\text{TM}}_{20}$.

The figures show that the standard phase I analysis method, $\text{ATM}_{\overline{\overline{X}}}$, performs almost as well as $\overline{\overline{X}}$ under normality when no contaminations are present and seems to be robust against localized variance disturbances. However, the method loses efficiency in the other situations. Because $\overline{\overline{X}}$, the initial estimate of μ, is highly sensitive to disturbances, the phase I limits are biased and fail to identify the correct out-of-control samples.

The mean rank method, denoted by ATM_{MR}, performs well under normality and when there are localized mean disturbances. The reason is that this estimator screens for samples with a mean rank significantly higher than that of the other samples. On the other hand, ATM_{MR} performs badly when diffuse outliers are present. The mean rank is not influenced by extreme occasional outliers so that samples containing only one outlier are not filtered out and hence are included in the calculation of the grand sample mean.

The third method, $\text{ATM}_{\overline{\text{TM}}_{20}}$, which uses the robust estimator $\overline{\text{TM}}_{20}$ to construct a phase I control chart, seems to be more efficient under normality than $\overline{\text{TM}}_{20}$ itself. The gain in efficiency can be explained by the use of an efficient estimator to obtain the final location estimate once screening is complete. Thus, an efficient phase I analysis does not require the use of an efficient estimator to construct the phase I control chart.

The final method, $\text{ATM}_{\overline{\text{TM}}'}$, which first screens for localized disturbances and then for occasional outliers, far outperforms all estimation methods. The method is particularly powerful in the presence of diffuse disturbances because its use of an individual control chart in phase I to identify single outliers increases the probability that such disturbances will be detected. For example, Figure 9 represents the situation where diffuse mean disturbances are present. The efficiency of the estimator improves for high b values because the disturbances are more likely to

FIGURE 6. MSE of Estimators when Symmetric Diffuse Variance Disturbances Are Present for $k = 30$. (a) $n = 5$, (b) $n = 9$.

FIGURE 7. MSE of Estimators when Asymmetric Diffuse Variance Disturbances Are Present for $k = 30$. (a) $n = 5$, (b) $n = 9$.

fall outside the control limits and are therefore more likely to be detected.

Derivation of the Phase II Control Limits

We now turn to the effect of the proposed location estimators on the $\overline{X}$ control-chart performance in phase II. The formulas for the $\overline{X}$ control limits with estimated limits are given by Equation (2). For the phase II control limits, we only estimate the in-control mean μ; we treat the in-control standard deviation σ as known because we want to isolate the effect of estimating the location parameter. The fac-

tor C_n that is used to obtain accurate control limits when the process parameters are estimated is derived such that the probability of a false signal equals the desired probability of a false signal. Except for the estimator $\overline{\overline{X}}$, C_n cannot be obtained easily in analytic form and is therefore obtained by means of simulation. The factors are chosen such that p from Equation (5) is equal to 0.0027 under normality. Fifty thousand simulation runs are used. For $k = 30$, $n = 5$, and $n = 9$, the resulting factors are equal to 3.05 for $\overline{\overline{X}}$, $\text{ATM}_{\overline{\overline{X}}}$, $\text{ATM}_{\overline{TM}_{20}}$, and $\text{ATM}_{\overline{TM}'}$; 3.06 for $\overline{\overline{X}}_{20}$ and $\overline{TM}$ and 3.07 for $M(\overline{X})$, $\overline{M}$, $\overline{HL}$, $\overline{TM}_{20}$ and $\text{ATM}_{\overline{MR}}$.
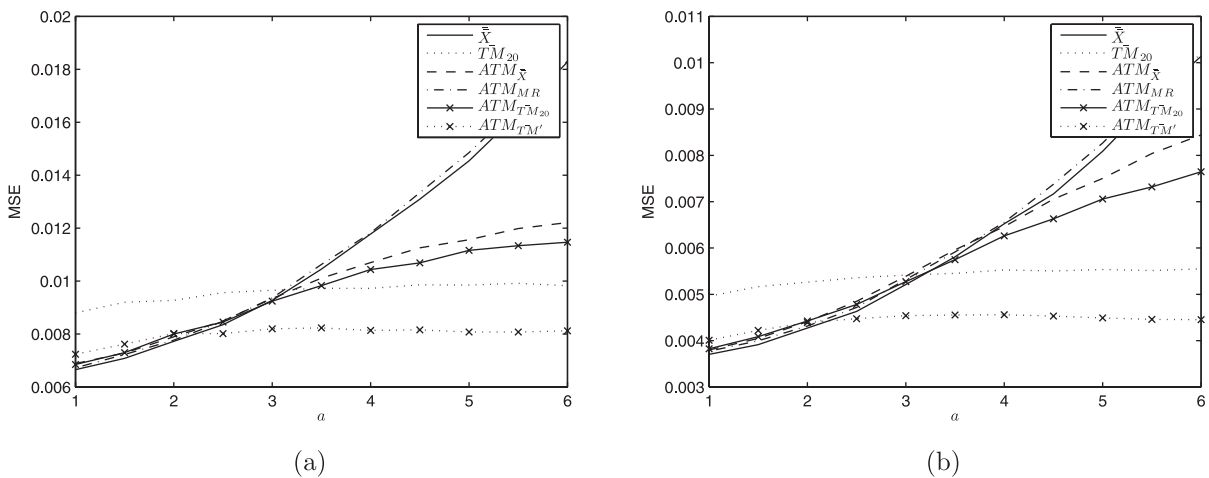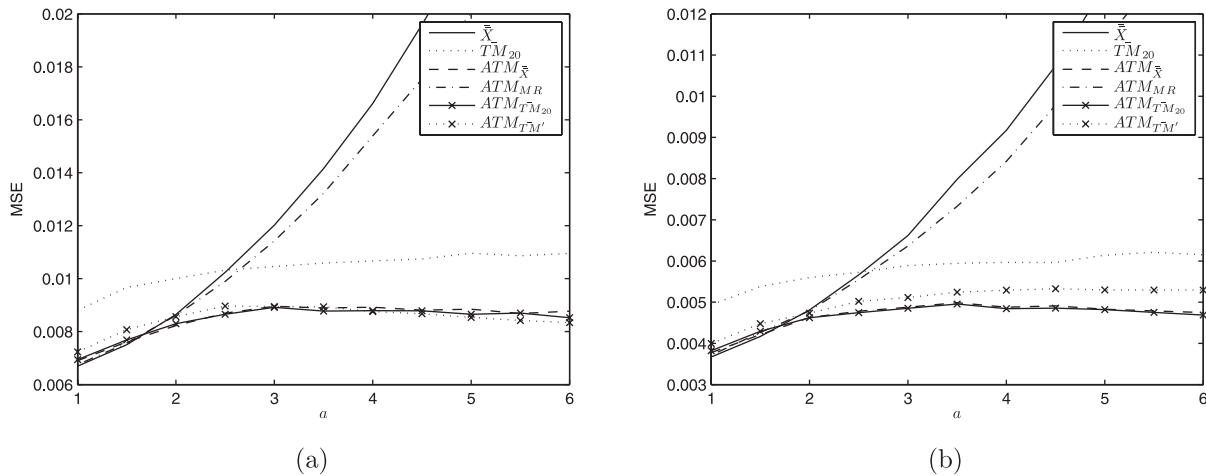

FIGURE 8. MSE of Estimators when Localized Variance Disturbances Are Present for $k = 30$. (a) $n = 5$, (b) $n = 9$.

(a)   (b)
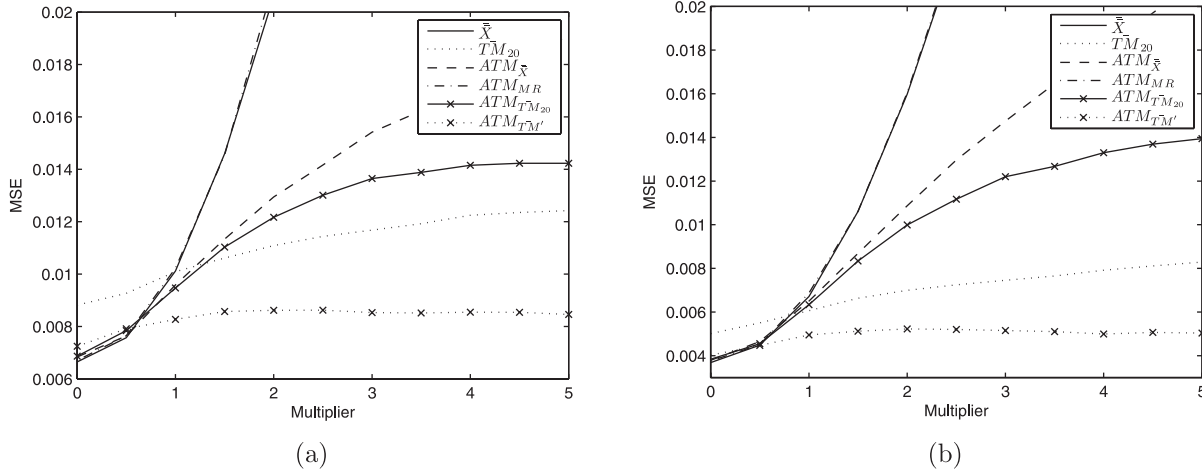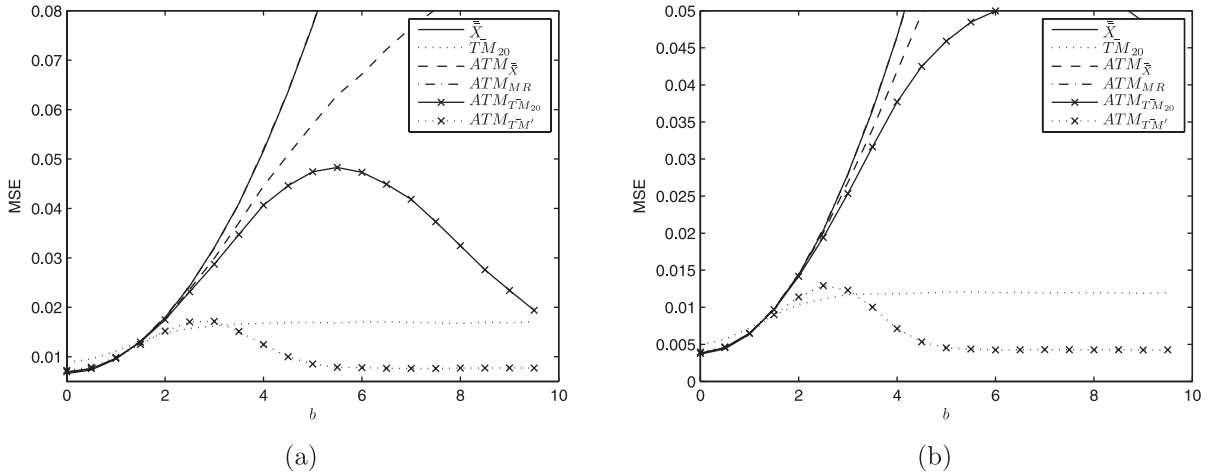
FIGURE 9. MSE of Estimators when Diffuse Mean Disturbances Are Present for $k = 30$. (a) $n = 5$, (b) $n = 9$.

Control-Chart Performance

In this section we evaluate the effect on $\overline{X}$ phase II performance of the proposed location statistics and estimation methods based on phase I control charting. We consider the same phase I situations as those used to assess the MSE with a, b and the multiplier equal to 4 to simulate the contaminated case (cf. the section entitled Efficiency of Proposed Estimators).

Following Jensen et al. (2006), we use the unconditional run-length distribution to assess performance. Specifically, we look at several characteristics of that distribution, namely the average run length (ARL) and the standard deviation of the run length (SDRL). In addition, we also give the probability that one sample gives a signal (p). We compute

these characteristics in an in-control and several out-of-control situations. We consider different shifts of size $\delta\sigma$ in the mean, setting δ equal to 0, 0.5, 1, and 2. The performance characteristics are obtained by simulation. The next section describes the simulation method, followed by the results for the control charts constructed in the uncontaminated situation and various contaminated situations.

Simulation Procedure

The performance characteristics p and ARL for estimated control limits are determined by averaging the conditional characteristics, i.e., the characteristics for a given set of estimated control limits, over all possible values of the control limits. Recall the definitions of $P(F_i \mid \hat{\mu}, \hat{\sigma})$ from Equation (3),

(a)   (b)

FIGURE 10. MSE of Estimators when Localized Mean Disturbances Are Present for $k = 30$. (a) $n = 5$, (b) $n = 9$.
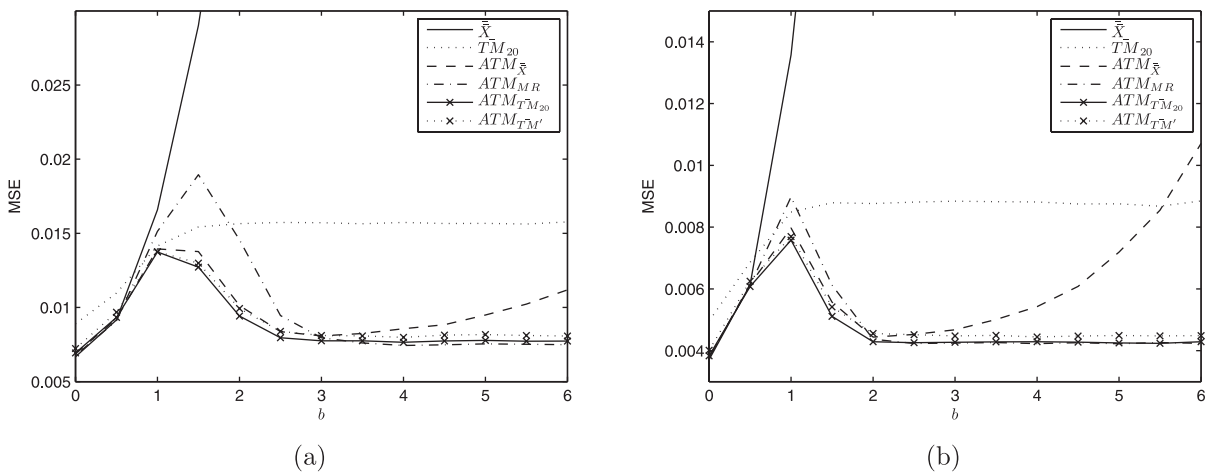
TABLE 3. p, ARL, and (in Parentheses) SDRL of Corrected Limits Under Normality and when
Symmetric Variance Disturbances Are Present for $k = 30$

| | n | $\hat{\mu}$ | p | | | | ARL and SDRL | | | |
			$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$
$N(0,1)$	5	$\overline{\overline{X}}$	0.0027	0.029	0.21	0.92	384 (392)	41.7 (49.4)	5.03 (4.90)	1.09 (0.32)
		$M(\overline{X})$	0.0027	0.028	0.21	0.91	390 (406)	46.2 (59.9)	5.31 (5.43)	1.10 (0.33)
		$\overline{M}$	0.0027	0.028	0.21	0.91	392 (407)	45.9 (59.0)	5.29 (5.37)	1.10 (0.33)
		$\overline{\overline{X}}_{20}$	0.0027	0.028	0.21	0.92	391 (401)	43.3 (52.4)	5.14 (5.08)	1.09 (0.32)
		$\overline{\text{HL}}$	0.0027	0.029	0.21	0.92	380 (389)	42.0 (50.4)	5.05 (4.94)	1.09 (0.32)
		$\overline{\text{TM}}$	0.0027	0.028	0.21	0.92	390 (400)	43.4 (53.0)	5.14 (5.09)	1.09 (0.32)
		$\overline{\text{TM}}_{20}$	0.0027	0.028	0.21	0.92	396 (410)	45.3 (56.9)	5.26 (5.29)	1.09 (0.33)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0027	0.029	0.21	0.92	383 (392)	41.8 (49.6)	5.04 (4.92)	1.09 (0.32)
		ATM_{MR}	0.0027	0.029	0.21	0.92	383 (391)	41.5 (49.1)	5.04 (4.89)	1.09 (0.32)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0027	0.029	0.21	0.92	382 (391)	41.8 (49.9)	5.04 (4.92)	1.09 (0.32)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0027	0.029	0.21	0.92	381 (390)	42.0 (50.3)	5.06 (4.96)	1.09 (0.32)
	9	$\overline{\overline{X}}$	0.0027	0.064	0.48	1.00	384 (393)	17.9 (20.0)	2.13 (1.62)	1.00 (0.043)
		$M(\overline{X})$	0.0027	0.063	0.47	1.00	390 (405)	19.5 (23.5)	2.19 (1.74)	1.00 (0.046)
		$\overline{M}$	0.0027	0.063	0.47	1.00	390 (405)	19.5 (23.6)	2.19 (1.74)	1.00 (0.046)
		$\overline{\overline{X}}_{20}$	0.0027	0.063	0.48	1.00	391 (401)	18.5 (21.1)	2.15 (1.66)	1.00 (0.044)
		$\overline{\text{HL}}$	0.0027	0.064	0.48	1.00	380 (389)	18.0 (20.4)	2.13 (1.63)	1.00 (0.043)
		$\overline{\text{TM}}$	0.0027	0.063	0.48	1.00	390 (400)	18.6 (21.4)	2.16 (1.67)	1.00 (0.045)
		$\overline{\text{TM}}_{20}$	0.0027	0.062	0.47	1.00	395 (409)	19.3 (22.7)	2.18 (1.71)	1.00 (0.046)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0027	0.064	0.48	1.00	382 (391)	17.9 (20.1)	2.13 (1.63)	1.00 (0.043)
		ATM_{MR}	0.0027	0.064	0.48	1.00	382 (391)	17.9 (20.1)	2.13 (1.63)	1.00 (0.043)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0027	0.064	0.48	1.00	382 (391)	18.0 (20.2)	2.13 (1.63)	1.00 (0.043)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0027	0.064	0.48	1.00	380 (390)	18.0 (20.5)	2.13 (1.64)	1.00 (0.043)
$N(0,1)$	5	$\overline{\overline{X}}$	0.0030	0.030	0.21	0.92	358 (375)	45.0 (60.9)	5.21 (5.44)	1.09 (0.33)
&		$M(\overline{X})$	0.0029	0.029	0.21	0.91	375 (395)	48.4 (67.7)	5.41 (5.77)	1.10 (0.34)
$N(0,4)$		$\overline{M}$	0.0028	0.029	0.21	0.91	387 (403)	46.5 (61.5)	5.34 (5.52)	1.10 (0.33)
(symm occ)		$\overline{\overline{X}}_{20}$	0.0028	0.029	0.21	0.92	376 (392)	45.1 (58.8)	5.23 (5.35)	1.09 (0.33)
		$\overline{\text{HL}}$	0.0029	0.029	0.21	0.92	370 (383)	43.2 (54.5)	5.13 (5.15)	1.09 (0.32)
		$\overline{\text{TM}}$	0.0027	0.029	0.21	0.92	384 (396)	44.2 (55.6)	5.19 (5.21)	1.09 (0.32)
		$\overline{\text{TM}}_{20}$	0.0027	0.028	0.21	0.91	390 (405)	46.0 (59.2)	5.29 (5.38)	1.10 (0.33)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0030	0.030	0.21	0.92	362 (378)	44.4 (58.8)	5.18 (5.33)	1.09 (0.32)
		ATM_{MR}	0.0030	0.030	0.21	0.92	357 (374)	45.1 (61.6)	5.21 (5.44)	1.09 (0.33)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0029	0.030	0.21	0.92	364 (379)	44.3 (58.3)	5.18 (5.31)	1.09 (0.32)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0028	0.029	0.21	0.92	375 (386)	42.7 (52.8)	5.09 (5.06)	1.09 (0.32)

TABLE 3. *Continued*

	n	$\hat{\mu}$	p				ARL and SDRL			
			$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$
$N(0,1)$	9	$\bar{\bar{X}}$	0.0030	0.066	0.48	1.00	358 (375)	19.1 (24.0)	2.17 (1.73)	1.00 (0.045)
&		$M(\bar{X})$	0.0030	0.065	0.47	1.00	371 (393)	20.6 (27.5)	2.23 (1.83)	1.00 (0.048)
$N(0,4)$		$\bar{M}$	0.0028	0.063	0.47	1.00	386 (403)	19.8 (24.5)	2.20 (1.75)	1.00 (0.047)
(symm occ)		$\bar{\bar{X}}_{20}$	0.0029	0.064	0.48	1.00	373 (389)	19.4 (24.1)	2.18 (1.74)	1.00 (0.046)
		$\overline{\mathrm{HL}}$	0.0028	0.064	0.48	1.00	373 (385)	18.4 (21.5)	2.14 (1.66)	1.00 (0.044)
		$\overline{\mathrm{TM}}$	0.0027	0.063	0.48	1.00	384 (397)	18.8 (22.1)	2.16 (1.69)	1.00 (0.045)
		$\overline{\mathrm{TM}}_{20}$	0.0027	0.063	0.47	1.00	391 (406)	19.4 (23.3)	2.19 (1.74)	1.00 (0.046)
		$\mathrm{ATM}_{\bar{\bar{X}}}$	0.0030	0.066	0.48	1.00	358 (375)	19.2 (24.0)	2.17 (1.73)	1.00 (0.046)
		ATM_{MR}	0.0031	0.066	0.48	1.00	356 (374)	19.1 (24.1)	2.17 (1.73)	1.00 (0.046)
		$\mathrm{ATM}_{\overline{\mathrm{TM}}_{20}}$	0.0030	0.066	0.48	1.00	360 (376)	19.0 (23.5)	2.16 (1.71)	1.00 (0.046)
		$\mathrm{ATM}_{\overline{\mathrm{TM}}'}$	0.0028	0.064	0.48	1.00	375 (386)	18.3 (21.3)	2.14 (1.65)	1.00 (0.044)

$E(\mathrm{RL} \mid \hat{\mu}, \hat{\sigma})$ from (4), $p = E(P(F_i \mid \hat{\mu}, \hat{\sigma}))$ from Equation (5), and $\mathrm{ARL} = E((1/P(F_i \mid \hat{\mu}, \hat{\sigma})))$ from Equation (6). These expectations will be obtained by simulation: numerous datasets are generated and, for each dataset, $P(F_i \mid \hat{\mu})$ and $E(\mathrm{RL} \mid \hat{\mu})$ are computed. Note that we take $\hat{\sigma} = \sigma$. By averaging these values, we obtain the unconditional values. The unconditional standard deviation is determined by

$$\begin{aligned}
&\mathrm{SDRL} \\
&= \sqrt{\mathrm{Var}(\mathrm{RL})} \\
&= \sqrt{E(\mathrm{Var}(\mathrm{RL} \mid \hat{\mu})) + \mathrm{Var}(E(\mathrm{RL} \mid \hat{\mu}))} \\
&= \left(2E\left(\frac{1}{P(F_i \mid \hat{\mu})}\right)^2 - \left(E\frac{1}{P(F_i \mid \hat{\mu})}\right)^2 \right. \\
&\quad \left. - E\frac{1}{P(F_i \mid \hat{\mu})} \right)^{1/2}.
\end{aligned}$$

Enough replications of the above procedure were performed to obtain sufficiently small relative estimated standard errors for p and ARL. The relative estimated standard error is the estimated standard error of the estimate relative to the estimate. The relative standard error of the estimates is never higher than 0.60%.

Results

First, we consider the situation where the process follows a normal distribution and the phase I data are not contaminated. We investigate the impact of the estimator used to estimate μ in phase I. Table 3 presents the probability of one sample showing a signal (p) and the average run length (ARL) when the process mean equals $\mu + \delta\sigma$. When $\delta = 0$, the process is in control, so we want p to be as low as possible and ARL to be as high as possible. When $\delta \neq 0$, i.e., in the out-of-control situation, we want to achieve the opposite. We can see that, in the absence of any contamination (first part of Table 3), the efficiency of the estimators is very similar. We can therefore conclude that using a more robust location estimator does not have a substantial impact on the control-chart performance in the uncontaminated situation.

The phase II control charts based on the estimation methods $M(\bar{X})$, $\bar{\bar{X}}_{20}$, $\overline{\mathrm{TM}}_{20}$, $\mathrm{ATM}_{\bar{\bar{X}}}$, and ATM_{MR} perform relatively well when localized disturbances are present, while the charts based on $\bar{M}$, $\overline{\mathrm{HL}}$, $\overline{\mathrm{TM}}$, and $\overline{\mathrm{TM}}_{20}$ perform relatively well when diffuse disturbances are present (cf., Tables 3–5).

The phase II chart based on $\mathrm{ATM}_{\overline{\mathrm{TM}}'}$ performs best: this chart is as efficient as $\bar{\bar{X}}$ in the uncontaminated normal situation and its performance does not change much when contaminations come into play. Moreover, the chart outperforms the other methods in all situations because it successfully filters out both diffuse and localized disturbances. In the presence of asymmetric disturbances, in particular, the added value of this estimation method is substantial.

TABLE 4. p, ARL and (in Parentheses) SDRL of Corrected Limits when
Asymmetric Variance or Localized Variance Disturbances Are Present for $k = 30$

	n	$\hat{\mu}$	p				ARL and SDRL			
			$\delta=0$	$\delta=0.5$	$\delta=1$	$\delta=2$	$\delta=0$	$\delta=0.5$	$\delta=1$	$\delta=2$
$N(0,1)$	5	$\overline{\overline{X}}$	0.0076	0.012	0.12	0.82	233 (295)	143 (210)	12.8 (24.0)	1.23 (0.60)
&		$M(\overline{X})$	0.0034	0.019	0.16	0.88	347 (378)	77.0 (110)	7.34 (8.26)	1.14 (0.41)
$N(0,4)$		$\overline{M}$	0.0029	0.022	0.18	0.90	374 (395)	61.6 (82.4)	6.32 (6.73)	1.12 (0.37)
(asymm occ)		$\overline{\overline{X}}_{20}$	0.0034	0.018	0.16	0.88	337 (366)	75.9 (103)	7.22 (7.91)	1.14 (0.41)
		$\overline{\text{HL}}$	0.0033	0.020	0.17	0.89	340 (363)	67.3 (90.4)	6.73 (7.36)	1.13 (0.39)
		$\overline{\text{TM}}$	0.0030	0.021	0.17	0.89	365 (384)	61.2 (69.2)	6.33 (6.61)	1.12 (0.37)
		$\overline{\text{TM}}_{20}$	0.0029	0.022	0.18	0.90	379 (398)	60.4 (78.7)	6.26 (6.56)	1.12 (0.37)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0034	0.020	0.17	0.89	334 (359)	70.0 (95.0)	6.86 (7.51)	1.13 (0.39)
		ATM_{MR}	0.0076	0.012	0.12	0.82	232 (294)	143 (209)	13.0 (26.7)	1.24 (0.61)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0032	0.021	0.17	0.89	347 (367)	62.9 (82.9)	6.46 (6.84)	1.12 (0.38)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0028	0.025	0.20	0.91	373 (385)	48.9 (60.2)	5.53 (5.55)	1.10 (0.34)
	9	$\overline{\overline{X}}$	0.011	0.020	0.27	0.99	175 (239)	89.5 (148)	4.58 (6.42)	1.01 (0.12)
		$M(\overline{X})$	0.0044	0.035	0.36	0.99	299 (346)	42.1 (63.4)	3.04 (2.89)	1.01 (0.077)
		$\overline{M}$	0.0031	0.048	0.42	1.00	366 (389)	26.8 (34.1)	2.51 (2.12)	1.00 (0.058)
		$\overline{\overline{X}}_{20}$	0.0047	0.033	0.35	0.99	280 (324)	42.7 (61.0)	3.09 (2.89)	1.01 (0.078)
		$\overline{\text{HL}}$	0.0033	0.044	0.41	1.00	336 (358)	27.9 (33.9)	2.57 (2.15)	1.00 (0.059)
		$\overline{\text{TM}}$	0.0031	0.046	0.42	1.00	356 (378)	26.4 (31.9)	2.51 (2.08)	1.00 (0.057)
		$\overline{\text{TM}}_{20}$	0.0030	0.047	0.42	1.00	368 (391)	26.8 (33.2)	2.52 (2.11)	1.00 (0.058)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0046	0.035	0.36	0.99	282 (322)	39.6 (57.4)	2.98 (2.76)	1.01 (0.074)
		ATM_{MR}	0.011	0.021	0.27	0.99	175 (240)	89.3 (148)	4.58 (6.41)	1.01 (0.12)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0039	0.039	0.38	1.00	307 (339)	33.3 (44.1)	2.77 (2.44)	1.00 (0.066)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0028	0.055	0.45	1.00	370 (383)	21.4 (24.9)	2.30 (1.83)	1.00 (0.050)
$N(0,1)$	5	$\overline{\overline{X}}$	0.0034	0.032	0.22	0.91	337 (361)	48.7 (73.0)	5.42 (6.09)	1.10 (0.34)
&		$M(\overline{X})$	0.0028	0.029	0.21	0.91	382 (400)	47.4 (64.2)	5.40 (5.66)	1.10 (0.33)
$N(0,4)$		$\overline{M}$	0.0037	0.033	0.22	0.91	335 (372)	57.2 (98.5)	5.90 (7.45)	1.11 (0.36)
(localized)		$\overline{\overline{X}}_{20}$	0.0028	0.029	0.21	0.92	382 (395)	44.3 (55.8)	5.21 (5.25)	1.09 (0.32)
		$\overline{\text{HL}}$	0.0035	0.032	0.22	0.91	332 (358)	50.0 (76.8)	5.46 (6.22)	1.10 (0.34)
		$\overline{\text{TM}}$	0.0035	0.032	0.22	0.91	338 (368)	52.3 (83.4)	5.62 (6.63)	1.10 (0.35)
		$\overline{\text{TM}}_{20}$	0.0028	0.029	0.21	0.91	387 (403)	46.6 (61.7)	5.36 (5.55)	1.10 (0.33)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0028	0.029	0.21	0.92	371 (382)	43.1 (53.7)	5.12 (5.13)	1.09 (0.32)
		ATM_{MR}	0.0033	0.031	0.22	0.92	342 (364)	47.7 (69.5)	5.34 (5.83)	1.10 (0.33)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0028	0.029	0.21	0.92	371 (384)	42.9 (53.8)	5.12 (5.13)	1.09 (0.32)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0028	0.029	0.21	0.92	372 (384)	43.0 (53.7)	5.11 (5.10)	1.09 (0.32)

TABLE 4. *Continued*

	n	$\hat{\mu}$	p				ARL and SDRL			
			$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$
$N(0,1)$	9	$\overline{\overline{X}}$	0.0034	0.068	0.48	1.00	337 (362)	20.4 (28.6)	2.21 (1.84)	1.00 (0.048)
&		$M(\overline{X})$	0.0028	0.064	0.47	1.00	381 (400)	19.9 (25.0)	2.21 (1.78)	1.00 (0.047)
$N(0,4)$		$\overline{M}$	0.0038	0.069	0.47	1.00	332 (370)	24.0 (40.4)	2.32 (2.12)	1.00 (0.054)
(localized)		$\overline{\overline{X}}_{20}$	0.0028	0.064	0.48	1.00	382 (395)	18.9 (22.3)	2.17 (1.69)	1.00 (0.045)
		$\overline{\text{HL}}$	0.0035	0.069	0.48	1.00	333 (359)	20.8 (30.1)	2.22 (1.88)	1.00 (0.049)
		$\overline{\text{TM}}$	0.0035	0.068	0.48	1.00	338 (368)	21.8 (32.7)	2.25 (1.94)	1.00 (0.051)
		$\overline{\text{TM}}_{20}$	0.0028	0.063	0.47	1.00	385 (402)	19.7 (24.2)	2.20 (1.76)	1.00 (0.047)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0028	0.065	0.48	1.00	372 (384)	18.4 (21.6)	2.15 (1.67)	1.00 (0.044)
		ATM_{MR}	0.0033	0.068	0.48	1.00	342 (364)	20.0 (27.2)	2.20 (1.81)	1.00 (0.048)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0028	0.065	0.48	1.00	372 (384)	18.4 (21.7)	2.15 (1.67)	1.00 (0.044)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0029	0.065	0.48	1.00	368 (381)	18.6 (22.2)	2.15 (1.68)	1.00 (0.045)

When localized mean disturbances are present, we see a strange phenomenon for the $\overline{X}$, $\overline{M}$, $\overline{\text{HL}}$, and $\overline{\text{TM}}$ charts: the in-control ARL is lower than the out-of-control ARL for $\delta = 0.5$. In other words, these charts are more likely to give a signal in the in-control situation than in the out-of-control situation for $\delta = 0.5$ and hence, in the presence of disturbances, are highly ARL-biased (cf., Jensen et al. (2006)).

Concluding Remarks

In this article, we have considered several phase I estimators of the location parameter for use in establishing phase II control limits of $\overline{\overline{X}}$ charts. The collection includes robust estimators proposed in the existing literature as well as several phase I analyses, which apply a control chart retrospectively to study a historical dataset. The MSE of the estimators has been assessed under various circumstances: the uncontaminated situation and various situations contaminated with diffuse symmetric and asymmetric variance disturbances, localized variance disturbances, diffuse mean disturbances, and localized mean disturbances. Moreover, we have investigated the effect of the location estimator on the $\overline{X}$ phase II control-chart performance when the methods are used to determine the phase II limits.

The standard methods suffer from a number of problems. Estimators that are based on the principle of trimming individual observations (e.g., $\overline{M}$ and $\overline{\text{TM}}$) perform reasonably well when there are diffuse disturbances but not when localized disturbances are present. In the latter situation, estimators that are based on the principle of trimming samples (e.g., $M(\overline{X})$ and $\overline{\overline{X}}_{20}$) are efficient. All of these methods are biased when there are asymmetric disturbances, as the trimming principle does not take into account the asymmetry of the disturbance.

A phase I analysis, using a control chart to study a historical dataset retrospectively and trim the data adaptively, does take into account the distribution of the disturbance and is therefore very suitable for use during the estimation of μ. However, the standard method based on the $\overline{\overline{X}}$ phase I control chart has certain limitations. First, the initial estimate, $\overline{\overline{X}}$, is very sensitive to outliers so that the phase I limits are biased. As a consequence, the wrong data samples are often filtered out. Second, the sample mean is usually plotted on the phase I control chart, which makes it difficult to detect outliers in individual observations. Moreover, deleting the entire sample instead of the single outlier reduces efficiency.

In this article, we have proposed a new type of phase I analysis that addresses the problems encountered in the standard phase I analysis. The initial estimate of μ for the phase I control chart is based on a trimmed version of the trimean, namely, $\overline{\text{TM}}_{20}$, and a subsequent procedure for both sample screening and outlier screening (resulting in $\text{ATM}_{\overline{\text{TM}}'}$). The proposed method is efficient under normality and far outperforms the existing methods when disturbances

TABLE 5. p, ARL and (in Parentheses) SDRL of Corrected Limits when Diffuse Mean or
Localized Mean Disturbances Are Present for $k = 30$

			p				ARL and SDRL			
	n	$\hat{\mu}$	$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$
$N(0,1)$	5	$\overline{\overline{X}}$	0.0061	0.011	0.11	0.83	224 (271)	137 (182)	10.9 (12.9)	1.22 (0.53)
&		$M(\overline{X})$	0.0048	0.014	0.13	0.85	289 (340)	115 (168)	9.57 (12.2)	1.19 (0.49)
$N(4,1)$		$\overline{M}$	0.0033	0.019	0.16	0.88	351 (380)	74.6 (103)	7.12 (7.89)	1.14 (0.41)
(asymm occ)		$\overline{\overline{X}}_{20}$	0.0049	0.013	0.13	0.85	271 (316)	115 (158)	9.52 (11.2)	1.19 (0.49)
		$\overline{\text{HL}}$	0.0042	0.015	0.14	0.86	290 (326)	93.0 (126)	8.22 (9.18)	1.16 (0.45)
		$\overline{\text{TM}}$	0.0035	0.017	0.15	0.88	333 (361)	77.8 (103)	7.33 (7.95)	1.14 (0.41)
		$\overline{\text{TM}}_{20}$	0.0032	0.019	0.16	0.88	356 (383)	72.5 (97.0)	7.00 (7.57)	1.13 (0.40)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0055	0.012	0.12	0.84	245 (392)	123 (167)	10.0 (12.0)	1.20 (0.51)
		ATM_{MR}	0.0061	0.011	0.11	0.83	224 (272)	136 (182)	10.9 (12.9)	1.21 (0.53)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0052	0.013	0.12	0.85	257 (302)	116 (159)	9.58 (11.2)	1.19 (0.49)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0031	0.024	0.19	0.90	356 (374)	57.0 (78.1)	6.01 (6.39)	1.11 (0.36)
	9	$\overline{\overline{X}}$	0.0089	0.018	0.26	0.99	161 (208)	77.3 (107)	4.17 (4.16)	1.01 (0.11)
		$M(\overline{X})$	0.0074	0.023	0.29	0.99	212 (274)	71.7 (111)	3.96 (4.17)	1.01 (0.10)
		$\overline{M}$	0.0035	0.041	0.39	1.00	339 (371)	32.4 (42.7)	2.72 (2.39)	1.00 (0.065)
		$\overline{\overline{X}}_{20}$	0.0075	0.021	0.28	0.99	193 (246)	68.9 (98.5)	3.91 (3.90)	1.01 (0.10)
		$\overline{\text{HL}}$	0.0046	0.033	0.35	0.99	272 (310)	39.6 (51.4)	3.00 (2.69)	1.01 (0.074)
		$\overline{\text{TM}}$	0.0037	0.038	0.38	1.00	317 (349)	33.6 (42.7)	2.79 (2.43)	1.00 (0.067)
		$\overline{\text{TM}}_{20}$	0.0035	0.039	0.38	1.00	336 (368)	32.9 (42.0)	2.75 (2.40)	1.00 (0.066)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0081	0.020	0.28	0.99	177 (227)	71.0 (99.9)	3.98 (3.96)	1.01 (0.10)
		ATM_{MR}	0.0089	0.019	0.26	0.99	162 (209)	77.8 (108)	4.16 (4.15)	1.01 (0.11)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0074	0.022	0.29	0.99	193 (242)	64.9 (91.2)	3.80 (3.73)	1.00 (0.098)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0031	0.051	0.43	1.00	352 (371)	24.5 (30.7)	2.42 (2.00)	1.00 (0.054)
$N(0,1)$	5	$\overline{\overline{X}}$	0.017	0.0034	0.046	0.70	72.3 (87.5)	329 (351)	25.0 (28.8)	1.45 (0.83)
&		$M(\overline{X})$	0.0031	0.021	0.17	0.89	366 (389)	66.0 (89.3)	6.62 (7.15)	1.13 (0.38)
$N(4,1)$		$\overline{M}$	0.017	0.0033	0.045	0.69	80.1 (105)	343 (372)	27.3 (33.6)	1.47 (0.86)
(localized)		$\overline{\overline{X}}_{20}$	0.0030	0.020	0.17	0.89	360 (379)	64.8 (81.8)	6.57 (6.81)	1.13 (0.38)
		$\overline{\text{HL}}$	0.017	0.0034	0.047	0.70	72.8 (89.3)	327 (350)	25.2 (29.2)	1.45 (0.83)
		$\overline{\text{TM}}$	0.017	0.0033	0.046	0.69	75.6 (94.4)	337 (362)	26.0 (30.7)	1.46 (0.84)
		$\overline{\text{TM}}_{20}$	0.0031	0.019	0.16	0.89	360 (385)	70.4 (92.0)	6.89 (7.32)	1.13 (0.40)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0028	0.026	0.20	0.91	373 (385)	47.5 (49.1)	5.44 (5.47)	1.10 (0.33)
		ATM_{MR}	0.0028	0.029	0.21	0.92	378 (388)	42.1 (50.9)	5.08 (4.99)	1.09 (0.32)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0028	0.029	0.21	0.92	378 (388)	42.8 (52.1)	5.12 (5.07)	1.09 (0.32)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0028	0.029	0.21	0.92	375 (386)	43.4 (53.4)	5.14 (5.11)	1.09 (0.32)

TABLE 5. *Continued*

				p			ARL and SDRL			
	n	$\hat{\mu}$	$\delta=0$	$\delta=0.5$	$\delta=1$	$\delta=2$	$\delta=0$	$\delta=0.5$	$\delta=1$	$\delta=2$
$N(0,1)$	9	$\overline{\overline{X}}$	0.035	0.0039	0.11	0.96	34.3 (30.7)	293 (321)	10.1 (10.7)	1.04 (0.22)
&		$M(\overline{X})$	0.0031	0.048	0.42	1.00	366 (389)	26.8 (34.3)	2.51 (2.12)	1.00 (0.058)
$N(4,1)$		$\overline{M}$	0.034	0.0039	0.11	0.95	37.9 (48.5)	308 (346)	10.8 (12.3)	1.05 (0.23)
(localized)		$\overline{\overline{X}}_{20}$	0.0030	0.046	0.41	1.00	359 (379)	26.3 (31.5)	2.51 (2.07)	1.00 (0.057)
		$\overline{\text{HL}}$	0.035	0.0039	0.11	0.96	34.5 (40.8)	293 (322)	10.1 (10.9)	1.05 (0.22)
		$\overline{\text{TM}}$	0.034	0.0039	0.11	0.96	35.7 (43.1)	301 (333)	10.4 (11.4)	1.05 (0.22)
		$\overline{\text{TM}}_{20}$	0.0031	0.044	0.41	1.00	361 (385)	28.4 (35.4)	2.58 (2.18)	1.00 (0.060)
		$\text{ATM}_{\overline{\overline{X}}}$	0.0029	0.055	0.45	1.00	366 (380)	21.7 (25.9)	2.30 (1.85)	1.00 (0.050)
		ATM_{MR}	0.0028	0.064	0.48	1.00	377 (387)	18.2 (20.9)	2.14 (1.64)	1.00 (0.044)
		$\text{ATM}_{\overline{\text{TM}}_{20}}$	0.0028	0.063	0.48	1.00	378 (388)	18.4 (21.1)	2.15 (1.66)	1.00 (0.044)
		$\text{ATM}_{\overline{\text{TM}}'}$	0.0028	0.063	0.48	1.00	376 (386)	18.6 (21.5)	2.16 (1.67)	1.00 (0.044)

are present. Consequently, $\text{ATM}_{\overline{\text{TM}}'}$ is a very effective method for estimating μ for the $\overline{X}$ phase II chart limits.

Acknowledgments

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