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The $\overline{X}$ Control Chart under Non-Normality

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This paper studies design schemes for the $\overline{X}$ control chart under non-normality. Different estimators of the standard deviation are considered and the effect of the estimator on the performance of the control chart under non-normality is investigated. Two situations are distinguished. In the first situation, the effect of non-normality on the $\overline{X}$ control chart is investigated by using the control limits based on normality. In the second situation we incorporate the knowledge of non-normality to correct the limits of the $\overline{X}$ control chart. The schemes are evaluated by studying the characteristics of the in-control and the out-of-control run length distributions. The results indicate that when the control limits based on normality are applied the best estimator is the pooled sample standard deviation both under normality and under non-normality. When the control limits are corrected for non-normality, the estimator based on Gini's mean sample differences is the best choice. Copyright © 2009 John Wiley & Sons, Ltd.

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Introduction

This paper studies the $\overline{X}$ control chart in the situation that the limits are estimated and the process distribution is not normal. Let Y_{ij}, $i=1,2,\ldots$ and $j=1,2,\ldots,n$ denote the jth observation in sample i. The classical Shewhart control chart assumes that Y_{ij} are $N(\mu+\delta\sigma,\sigma^2)$ distributed, where μ and σ are known and δ is a constant. When $\delta=0$ the process is in-control, otherwise the process is shifted. The mean of this process can be monitored by plotting the sample means $\bar{Y}_i=1/n\sum_{j=1}^{n} Y_{ij}$ on the Shewhart control chart with upper control limit (UCL) and lower control limit (LCL):

$$UCL=\mu+3\frac{\sigma}{\sqrt{n}},\quad LCL=\mu-3\frac{\sigma}{\sqrt{n}} \tag{1}$$

When $\bar{Y}_i$ is beyond the limits the process is considered to be out-of-control. Define RL_δ as the run length, that is the number of samples until the first sample mean is beyond the limits, when the process mean equals $\mu+\delta\sigma$. The performance of a control chart can be assessed by studying the characteristics of RL_δ for different values of δ. Two functions of interest are the probability of showing a signal in one sample (P_δ) and the average run length (ARL_δ). When the classical Shewhart control limits are applied (cf. (1)) and the assumptions are met, RL_δ is geometrically distributed. P_δ is given by $1-\Phi(3-\delta\sqrt{n})+\Phi(-3-\delta\sqrt{n})$, where Φ denotes the standard normal distribution and ARL_δ can be obtained by $1/P_\delta$. From the preceding we can derive the performance characteristics in the in-control situation: $P_0=0.0027$ and $ARL_0=370.4$.

When μ and σ are unknown, the limits need to be estimated. Woodall and Montgomery[1] define this phase as Phase I. They define the monitoring phase as Phase II. Estimating the parameters has two consequences for the performance of the control chart in Phase II. First, when the parameters are estimated and the estimations are simply plugged into (1), P_0 will deviate from the 0.0027 intended. Second, the run length distribution is no longer geometric. The latter issue is first addressed by Quesenberry[2]. Quesenberry argues that the number of estimation samples k should be at least $400/(n-1)$ in order to get limits that behave like known limits. This is of course unrealistic in most practical situations where we usually have 20–30 subgroups of sizes around 3–10 (see e.g. Ryan[3] and Montgomery[4]). In order to get accurate limits for moderate sample sizes, one could consider factors that replace the fixed constant 3 in (1). Another option is to investigate the influence of the estimator of the standard deviation. Schoonhoven *et al.*[5] study design schemes for the $\overline{X}$ control chart under normality. Different estimators of the standard deviation are considered and for each scheme the correction factor is derived by controlling P_0. They conclude that the control chart based on the pooled sample standard deviation is the best option under normality.

In practice, the normality assumption is often violated. Alwan and Roberts[6] examine 235 quality control applications and find that in most cases the assumptions of normality and independence are not fulfilled, resulting in incorrect control limits. The impact of non-normality on the performance of the control chart can be substantial. Shewhart[7] shows that the probability of false signalling of

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the $\overline{X}$ control chart with 3σ limits is smaller than 0.11 irrespective of the underlying distribution, and smaller than 0.05 for distributions likely to be encountered in practice, i.e. for strongly unimodal distributions. Schilling and Nelson[8] study the performance of the $\overline{X}$ control chart with limits conform (1). They conclude that the sample size n should be at least 4 in order to assure P_0 to be 0.014 or less. Padgett *et al.*[9] examine the impact of non-normality on the design scheme in (1) when μ and σ are estimated by their usual estimators, i.e. for μ the mean of the sample means and for σ the mean sample standard deviation or the mean sample range. They also conclude that the in-control probability of signalling of both charts greatly increases under non-normality. Several researchers correct the control limits based on the shape of the underlying distribution. Burr[10] studies the effect of non-normality on the $\overline{X}$ control chart considering various degrees of skewness and kurtosis. He determines constants for each degree of non-normality. Albers and Kallenberg[11] use the normal power family to model the underlying distribution.

This paper studies design schemes for the $\overline{X}$ control chart under non-normality in a different way. We propose different estimators of the standard deviation and study the effect of the estimator on the control chart performance under non-normality for moderate sample sizes (20 subgroups of sizes 4–10). Two situations are distinguished. In the first situation, the effect of non-normality on the $\overline{X}$ control chart is investigated by using the control limits based on normality. In the second situation, we incorporate the knowledge of non-normality to correct the limits of the $\overline{X}$ control chart. This approach is similar to the type of approach applied by Burr[10] and Albers and Kallenberg[11], where the control limits based on the usual estimators of the standard deviation, i.e. the mean sample standard deviation and the mean sample range, are also corrected for non-normality. In this paper we also consider other estimators of the standard deviation, such as the pooled sample standard deviation, Gini's mean sample differences and the mean sample interquartile range. In Albers and Kallenberg[11] a distinction is made between the model error and the stochastic error. The model error is defined as the error due to the incorrect distributional assumption and the stochastic error is defined as the error due to estimation. Comparing these two types of errors to the situations described above, in the first situation both the model and the stochastic error are involved, whereas in the second situation only the stochastic error is present. To investigate the effect of non-normality on the design schemes, we consider two cases: one by disturbing the kurtosis, i.e. the peak and tail behavior of the distribution, and the other by disturbing the skewness, i.e. the symmetry of the distribution. The simulations are performed to study the in-control and the out-of-control run length distributions.

The paper is organized as follows. The next section presents the design schemes, including the estimators that are applied and the determination of the control limits. In the subsequent section the schemes are evaluated by the use of simulation. The paper ends with concluding remarks.

Design schemes

In this study we investigate the effect of non-normality on the design scheme

$$\widehat{UCL}=\hat{\mu}+c(n,k,1-p/2)\frac{\hat{\sigma}}{\sqrt{n}},\quad \widehat{LCL}=\hat{\mu}+c(n,k,p/2)\frac{\hat{\sigma}}{\sqrt{n}} \tag{2}$$

where a hat above an alphabet represents an estimator and $c(n,k,1-p/2)$ and $c(n,k,p/2)$ denote the factors that are dependent on the number of samples k, the sample size n and p, the latter being equal to P_0. In this section we present the estimators of μ and σ that are considered and the determination of the factors.

Let X_{ij}, $i=1,2,\ldots,k$ and $j=1,2,\ldots,n$ denote the Phase I data and let Y_{ij}, $i=1,2,\ldots$ and $j=1,2,\ldots,n$ denote the Phase II data. We assume that X_{ij} are independent and identically distributed with mean μ and standard deviation σ and that Y_{ij} are independent and distributed according to the same type of distribution as X_{ij}, with the only difference that the mean can be shifted to $\mu+\delta\sigma$. In the study we distinguish two situations. In the first situation, we study the effect of non-normality on the $\overline{X}$ control chart with limits based on normality. Thus, X_{ij} and Y_{ij} are incorrectly assumed to be normally distributed. In the second situation, we correct the limits for non-normality. We assume that the shape of the underlying distribution of X_{ij} and Y_{ij} is known, up to the location and scale parameter. The design schemes that are considered in this study are location and scale invariant. Therefore, the constants used to obtain unbiased estimators and the factors that are applied for the control limits can be corrected for non-normality.

To investigate the effect of non-normality on the resulting schemes, we consider two cases: one by disturbing the kurtosis and the other by disturbing the symmetry of the distribution. For the case of disturbance in the kurtosis we use the Student's t distribution with 4 and 10 degrees of freedom and the logistic distribution, and for the disturbance in the symmetry we use the exponential distribution and the chi-squared distribution with 5 and 20 degrees of freedom. Note that the results for the exponential distribution are independent of the parameter value of the exponential distribution since this parameter only influences the scale of the distribution.

Estimators of spread

We estimate the process mean μ by the unbiased estimator

$$\bar{\bar{X}}=\frac{1}{k}\sum_{i=1}^{k}\left(\frac{1}{n}\sum_{j=1}^{n}X_{ij}\right) \tag{3}$$

i.e. the *grand sample mean*. The primary issue is the choice of the estimator of σ. We consider several estimators of σ. Below, the statistics and for each statistic the constant by which the statistic has to be divided in order to obtain an unbiased estimator of σ under normality are given. These constants are relevant to the first situation described.

The first estimator of σ that we consider is based on the *pooled sample standard deviation*

$$\tilde{S}=\left(\frac{1}{k}\sum_{i=1}^{k}S_i^2\right)^{1/2} \tag{4}$$

where S_i is the ith sample standard deviation defined by

$$S_i=\left(\frac{1}{n-1}\sum_{j=1}^{n}(X_{ij}-\bar{X}_i)^2\right)^{1/2}$$

An unbiased estimator of σ is $\tilde{S}/c_4(k(n-1)+1)$, where $c_4(m)$ is defined by

$$c_4(m)=\left(\frac{2}{m-1}\right)^{1/2}\frac{\Gamma(m/2)}{\Gamma((m-1)/2)}$$

Another unbiased estimator of σ is $\bar{S}/c_4(n)$, where $\bar{S}$ is the *mean sample standard deviation*

$$\bar{S}=\frac{1}{k}\sum_{i=1}^{k}S_i \tag{5}$$

We also consider the estimator based on the *mean sample range*

$$\bar{R}=\frac{1}{k}\sum_{i=1}^{k}R_i \tag{6}$$

where R_i is the range of the ith sample. We estimate σ by the unbiased estimator $\bar{R}/d_2(n)$, where $d_2(n)$ is the expected range of a random $N(0,1)$ sample of size n. Values of $d_2(n)$ can be found in Duncan[12], Table M.

The next estimator we propose is based on *Gini's mean sample differences*. Gini's mean differences of sample i are defined by

$$G_i=\sum_{j=1}^{n-1}\sum_{l=j+1}^{n}|X_{ij}-X_{il}|/(n(n-1)/2)$$

An unbiased estimator of σ is given by $\bar{G}/d_2(2)$, where

$$\bar{G}=\frac{1}{k}\sum_{i=1}^{k}G_i \tag{7}$$

The last estimator that we consider is based on the *mean sample interquartile range*. The interquartile range for sample i is defined by

$$IQR_i=Q_{75,i}-Q_{25,i}$$

where $Q_{r,i}$ is the rth percentile of the values in sample i. For a sample of size n, the sorted values $X_{(j),i}$, $j=1,2,\ldots,n$ denote the $P_{(j),i}=100(j-0.5)/n$ percentiles. Linear interpolation is used to compute the intermediate percentiles. For example, for a sample of size 5 the sorted values denoted by $X_{(1),i}, X_{(2),i}, X_{(3),i}, X_{(4),i}$ and $X_{(5),i}$ are, respectively, the 10, 30, 50, 70 and 90 percentiles. Then, $Q_{25,i}$ can be obtained by $X_{(1),i}+[(25-P_{(1),i})/(P_{(2),i}-P_{(1),i})](X_{(2),i}-X_{(1),i})$. In Kimball[13] it is shown that the best choice for the $P_{(j),i}$ would be $100(j-3/8)/(n+1/4)$ instead of $100(j-0.5)/n$. We could also have used the definition that for a sample of size n, the sorted values are the $100j/n, j=1,2,\ldots,n$ percentiles. However, the choice $100(j-0.5)/n$ is more intuitively and better known (cf. Madansky[14]). The unbiased estimator is $\overline{IQR}/q(n)$ where

$$\overline{IQR}=\frac{1}{k}\sum_{i=1}^{k}IQR_i \tag{8}$$

and $q(n)$ is defined as the expectation of the interquartile range of a random sample of n $N(0,1)$ distributed variables. Values of $q(n)$ can be derived from the mean positions of ranked normal deviates, which are given in Table 28 in Pearson and Hartley[15].

When the underlying distribution is not normal the constants $c_4(m)$, $d_2(n)$ and $q(n)$ are different. In the second situation that we consider, we incorporate the knowledge of non-normality to correct the limits. Therefore, the constants are corrected in this situation. The corrected constants are determined such that the expected value of the statistic divided by the constant is equal to the true value of σ. For example, for the estimator $\tilde{S}$, the new constant $c_4(m)$ is determined such that $E(\tilde{S})/c_4(m)=\sigma$. We obtain $E(\tilde{S})$ by simulation: we generate 100 000 times k samples of size n, compute $\tilde{S}$ for each instance and take the average of the values. The resulting constants are presented in Table I for $k=20$ and $n=4,6,8,10$. For comparison purposes the original values based on the normality assumption are also given in Table I. It follows that the differences between the normal case and the non-normal case can be substantial. As was to be expected, the largest differences with respect to the constants based on the normal distribution are shown by the t_4, exponential and χ^2_5 distribution. Rather small differences are shown by the t_{10}, logistic and χ^2_{20} distribution.

Table I. Constants to obtain unbiased estimators for σ for $k=20$

n	$\hat{\sigma}$	Normal	t_4	t_{10}	Logistic	Exponential	χ^2_5	χ^2_{20}
4	$\tilde{S}$	0.996	0.984	0.995	0.994	0.988	0.992	0.995
	$\bar{S}$	0.922	0.858	0.907	0.903	0.840	0.886	0.913
	$\bar{R}$	2.059	1.920	2.030	2.021	1.834	1.961	2.035
	$\bar{G}$	1.129	1.042	1.109	1.102	1.000	1.073	1.115
	$\overline{IQR}$	1.327	1.205	1.297	1.286	1.167	1.258	1.310
6	$\tilde{S}$	0.998	0.987	0.997	0.996	0.992	0.995	0.997
	$\bar{S}$	0.952	0.895	0.940	0.936	0.885	0.923	0.944
	$\bar{R}$	2.534	2.421	2.521	2.518	2.284	2.426	2.506
	$\bar{G}$	1.128	1.041	1.109	1.103	1.000	1.074	1.114
	$\overline{IQR}$	1.284	1.062	1.218	1.194	1.084	1.202	1.262
8	$\tilde{S}$	0.998	0.991	0.998	0.998	0.994	0.997	0.998
	$\bar{S}$	0.965	0.915	0.955	0.952	0.909	0.941	0.959
	$\bar{R}$	2.848	2.784	2.854	2.860	2.594	2.735	2.818
	$\bar{G}$	1.128	1.041	1.109	1.103	1.001	1.074	1.114
	$\overline{IQR}$	1.325	1.094	1.256	1.232	1.118	1.241	1.304
10	$\tilde{S}$	0.998	0.993	0.998	0.998	0.996	0.997	0.998
	$\bar{S}$	0.972	0.928	0.964	0.961	0.924	0.952	0.967
	$\bar{R}$	3.077	3.073	3.107	3.119	2.830	2.965	3.048
	$\bar{G}$	1.128	1.042	1.109	1.102	1.000	1.074	1.114
	$\overline{IQR}$	1.312	1.060	1.235	1.205	1.093	1.223	1.290

Determination of the control limits

In order to control the risk of having false alarms, the fixed constant 3, which is applied for the limits when the process distribution is normal and the parameters are known (cf. (1)), is replaced by the factors $c(n,k,1-p/2)$ and $c(n,k,p/2)$ in (2). Since the run length distribution is not geometric when the parameters are estimated, we should make in advance a decision on the purpose of the control chart. For example, should the chart perform well in terms of P_0, in terms of ARL_0 or in terms of a specific percentile point of the in-control run length distribution? Albers and Kallenberg[16] describe different correction methods for the $\bar{X}$ control chart. In this study we choose to take P_0 as a point of departure, i.e. we determine the factors $c(n,k,1-p/2)$ and $c(n,k,p/2)$ such that

$$P(\bar{Y}_i \le \widehat{LCL}) = P\left(\sqrt{n}\frac{\bar{Y}_i-\hat{\mu}}{\hat{\sigma}} \le c(n,k,p/2)\right) = p/2 \quad \text{and}$$
$$P(\bar{Y}_i \ge \widehat{UCL}) = P\left(\sqrt{n}\frac{\bar{Y}_i-\hat{\mu}}{\hat{\sigma}} \ge c(n,k,1-p/2)\right) = p/2 \qquad (9)$$

where $\bar{Y}_i$ is supposed to be in-control and p is chosen to be equal to 0.0027. The factors applied for the limits based on normality, relevant to the first situation that is considered, are derived analytically. For this derivation we refer to Schoonhoven *et al.*[5]. The factors applied in the second situation are chosen such that (9) holds under non-normality, where $P(\bar{Y}_i \le \widehat{LCL})$ and $P(\bar{Y}_i \ge \widehat{UCL})$ are obtained by simulation. The simulation procedure is described below.

Let E_i denote the event that the ith sample mean is beyond the limits. Further, denote by $P(E_i|\bar{\bar{X}},\hat{\sigma})$ the conditional probability that for given $\bar{\bar{X}}$ and $\hat{\sigma}$, the sample mean $\bar{Y}_i$ is beyond the control limits

$$P(E_i|\bar{\bar{X}},\hat{\sigma}) = P(\bar{Y}_i < \widehat{LCL} \text{ or } \bar{Y}_i > \widehat{UCL})$$

Given $\bar{\bar{X}}$ and $\hat{\sigma}$, the events E_s and E_t $(s \neq t)$ are independent. Therefore, the run length has a geometric distribution with parameter $P(E_i|\bar{\bar{X}},\hat{\sigma})$. When we take the expectation over the estimation data X_{ij} we get the unconditional probability of one sample showing a false alarm

$$P(E_i) = E(P(E_i|\bar{\bar{X}},\hat{\sigma}))$$

and, similarly, the unconditional *ARL*

$$ARL = E(1/P(E_i|\bar{\bar{X}},\hat{\sigma}))$$

These expectations are simulated by generating 10 000 times k data samples of size n, computing for each data set the conditional value and averaging the conditional values over the data sets. Note that for the calculation of the control limits in Phase I the process is considered to be in-control, thus outliers are omitted in this phase. Table II shows the factors for $k=20$ and $n=6$.

Table II. Factors for the $\overline{X}$ control chart for $n=6$ and $k=20$

					Exponential		χ^2_5		χ^2_{20}	
$\hat{\sigma}$	Normal	t_4	t_{10}	Logistic	Up	Low	Up	Low	Up	Low
$\tilde{S}$	3.145	3.950	3.282	3.310	4.530	2.098	3.984	2.441	3.547	2.788
$\bar{S}$	3.145	3.859	3.274	3.300	4.494	2.063	3.971	2.427	3.544	2.786
$\bar{R}$	3.145	3.869	3.270	3.310	4.490	2.063	3.973	2.432	3.546	2.788
$\bar{G}$	3.145	3.849	3.275	3.300	4.464	2.038	3.958	2.419	3.541	2.783
$\overline{IQR}$	3.225	3.884	3.342	3.369	4.560	2.130	4.040	2.486	3.621	2.863

Table III. P_δ of limits based on normality for $k=20$ and $n=6$

		P_δ for $\hat{\sigma}$ unbiased under normality				
	$\hat{\sigma}$	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=2$
Normal	$\tilde{S}$	0.0027	0.0081	0.034	0.25	0.95
	$\bar{S}$	0.0027	0.0081	0.034	0.25	0.95
	$\bar{R}$	0.0028	0.0081	0.034	0.25	0.95
	$\bar{G}$	0.0027	0.0081	0.034	0.25	0.95
	$\overline{IQR}$	0.0027	0.0077	0.032	0.24	0.94
t_4	$\tilde{S}$	0.0088	0.015	0.041	0.27	0.95
	$\bar{S}$	0.0105	0.018	0.050	0.31	0.97
	$\bar{R}$	0.0098	0.017	0.047	0.29	0.96
	$\bar{G}$	0.0113	0.019	0.054	0.33	0.97
	$\overline{IQR}$	0.0173	0.030	0.081	0.41	0.98
t_{10}	$\tilde{S}$	0.0038	0.0093	0.035	0.25	0.95
	$\bar{S}$	0.0041	0.0100	0.037	0.26	0.96
	$\bar{R}$	0.0040	0.0096	0.036	0.26	0.95
	$\bar{G}$	0.0043	0.0103	0.038	0.27	0.96
	$\overline{IQR}$	0.0053	0.0124	0.044	0.28	0.96
Logistic	$\tilde{S}$	0.0040	0.0098	0.036	0.25	0.95
	$\bar{S}$	0.0045	0.0108	0.039	0.27	0.96
	$\bar{R}$	0.0042	0.0102	0.037	0.26	0.95
	$\bar{G}$	0.0047	0.0112	0.040	0.27	0.96
	$\overline{IQR}$	0.0064	0.0145	0.049	0.30	0.96
Exponential	$\tilde{S}$	0.010	0.026	0.061	0.26	0.95
	$\bar{S}$	0.013	0.033	0.077	0.31	0.98
	$\bar{R}$	0.015	0.037	0.085	0.33	0.98
	$\bar{G}$	0.016	0.039	0.089	0.35	0.99
	$\overline{IQR}$	0.018	0.044	0.100	0.37	0.99
χ^2_5	$\tilde{S}$	0.0064	0.019	0.052	0.26	0.96
	$\bar{S}$	0.0073	0.021	0.058	0.28	0.97
	$\bar{R}$	0.0078	0.023	0.062	0.29	0.97
	$\bar{G}$	0.0080	0.023	0.063	0.29	0.97
	$\overline{IQR}$	0.0082	0.023	0.063	0.29	0.97
χ^2_{20}	$\tilde{S}$	0.0037	0.013	0.043	0.25	0.96
	$\bar{S}$	0.0039	0.014	0.044	0.26	0.96
	$\bar{R}$	0.0041	0.014	0.045	0.26	0.96
	$\bar{G}$	0.0040	0.014	0.045	0.26	0.96
	$\overline{IQR}$	0.0040	0.014	0.043	0.25	0.95

Table IV. ARL_δ of limits based on normality for $k=20$ and $n=6$

	$\hat{\sigma}$	ARL_δ for $\hat{\sigma}$ unbiased under normality				
		$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=2$
Normal	$\tilde{S}$	682	265	51.0	4.68	1.05
	$\bar{S}$	701	270	51.6	4.70	1.05
	$\bar{R}$	720	274	52.5	4.73	1.05
	$\bar{G}$	699	269	51.7	4.68	1.05
	$\overline{IQR}$	1474	509	81.3	5.11	1.07
t_4	$\tilde{S}$	202	162	68.5	11.8	1.16
	$\bar{S}$	128	86.1	33.6	4.39	1.04
	$\bar{R}$	140	94.7	37.2	4.76	1.04
	$\bar{G}$	114	75.6	29.1	3.89	1.03
	$\overline{IQR}$	77.6	50.0	19.1	2.88	1.02
t_{10}	$\tilde{S}$	470	219	53.0	4.93	1.05
	$\bar{S}$	409	193	45.2	4.65	1.05
	$\bar{R}$	451	212	48.9	4.85	1.05
	$\bar{G}$	391	185	43.6	4.55	1.05
	$\overline{IQR}$	437	203	46.5	4.68	1.05
Logistic	$\tilde{S}$	442	208	49.2	4.95	1.05
	$\bar{S}$	378	179	43.5	4.59	1.05
	$\bar{R}$	430	201	48.1	4.85	1.05
	$\bar{G}$	356	168	41.4	4.46	1.04
	$\overline{IQR}$	350	164	40.3	4.30	1.04
Exponential	$\tilde{S}$	251	90.4	34.3	6.17	1.06
	$\bar{S}$	160	58.8	23.1	4.53	1.03
	$\bar{R}$	136	50.5	20.1	4.06	1.02
	$\bar{G}$	122	45.6	18.3	3.79	1.02
	$\overline{IQR}$	117	43.5	17.4	3.65	1.02
χ^2_5	$\tilde{S}$	403	118	36.7	5.45	1.05
	$\bar{S}$	319	94.7	30.3	4.77	1.04
	$\bar{R}$	292	87.7	28.3	4.54	1.03
	$\bar{G}$	276	83.1	27.0	4.41	1.03
	$\overline{IQR}$	332	97.5	31.0	4.84	1.04
χ^2_{20}	$\tilde{S}$	620	163	41.3	5.04	1.05
	$\bar{S}$	571	153	39.1	4.88	1.05
	$\bar{R}$	561	151	38.7	4.84	1.04
	$\bar{G}$	545	147	37.9	4.79	1.04
	$\overline{IQR}$	939	211	50.5	5.68	1.06

Evaluation

In this section the design schemes are evaluated. The performance of the schemes is measured in terms of the probability of showing a signal in one sample (P_δ) and the average run length (ARL_δ) for the in-control situation ($\delta=0$) and several out-of-control situations ($\delta=0.25, 0.5, 1, 2$). We use the simulation method introduced in the previous paragraph to obtain these performance measures. The simulations are performed for six non-normal distribution functions: Student's t with 4 and 10 degrees of freedom, logistic, exponential and chi-squared with 5 and 20 degrees of freedom. The first paragraph of this section presents the results of the simulations when the limits based on normality are applied and the second paragraph shows the results for the case that the limits are corrected for non-normality.

Limits based on normality

In this paragraph we study the effect of non-normality on the $\overline{X}$ control chart with control limits based on the assumption of normality. This question is inspired by the fact that, according to the central limit theorem, the distribution of the sample means will approach normality for large sample sizes. Schilling and Nelson[8] show that the sample size n should be at least 4 in order to assure P_0 to be 0.014 or less when the process distribution is not normal. We investigate the effect of non-normality on estimated limits, and consider different estimators of the standard deviation. We present the results of the simulation for $n=6$ and $k=20$. Tables III and IV show the

Table V. P_δ of corrected limits for $k=20$ and $n=6$

		P_δ for unbiased $\hat{\sigma}$				
	$\hat{\sigma}$	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=2$
Normal	$\tilde{S}$	0.0027	0.0081	0.034	0.25	0.95
	$\bar{S}$	0.0027	0.0081	0.034	0.25	0.95
	$\bar{R}$	0.0028	0.0081	0.034	0.25	0.95
	$\bar{G}$	0.0027	0.0081	0.034	0.25	0.95
	$\overline{IQR}$	0.0027	0.0077	0.032	0.24	0.94
t_4	$\tilde{S}$	0.0027	0.0039	0.010	0.087	0.81
	$\bar{S}$	0.0027	0.0041	0.011	0.092	0.84
	$\bar{R}$	0.0027	0.0041	0.011	0.092	0.83
	$\bar{G}$	0.0027	0.0041	0.011	0.092	0.84
	$\overline{IQR}$	0.0027	0.0041	0.011	0.090	0.83
t_{10}	$\tilde{S}$	0.0027	0.0068	0.027	0.21	0.94
	$\bar{S}$	0.0027	0.0068	0.027	0.21	0.94
	$\bar{R}$	0.0027	0.0068	0.027	0.21	0.94
	$\bar{G}$	0.0027	0.0068	0.027	0.21	0.94
	$\overline{IQR}$	0.0027	0.0066	0.025	0.20	0.93
Logistic	$\tilde{S}$	0.0027	0.0066	0.025	0.20	0.93
	$\bar{S}$	0.0027	0.0067	0.025	0.21	0.93
	$\bar{R}$	0.0027	0.0067	0.025	0.20	0.93
	$\bar{G}$	0.0027	0.0067	0.025	0.21	0.93
	$\overline{IQR}$	0.0027	0.0066	0.024	0.19	0.92
Exponential	$\tilde{S}$	0.0027	0.0037	0.0098	0.057	0.60
	$\bar{S}$	0.0027	0.0037	0.0099	0.057	0.61
	$\bar{R}$	0.0027	0.0037	0.0099	0.057	0.61
	$\bar{G}$	0.0027	0.0037	0.0099	0.058	0.62
	$\overline{IQR}$	0.0027	0.0037	0.0097	0.056	0.59
χ^2_5	$\tilde{S}$	0.0027	0.0044	0.014	0.094	0.78
	$\bar{S}$	0.0027	0.0044	0.014	0.094	0.79
	$\bar{R}$	0.0027	0.0044	0.014	0.095	0.79
	$\bar{G}$	0.0027	0.0044	0.014	0.095	0.80
	$\overline{IQR}$	0.0027	0.0044	0.014	0.090	0.76
χ^2_{20}	$\tilde{S}$	0.0027	0.0058	0.021	0.16	0.90
	$\bar{S}$	0.0027	0.0059	0.021	0.16	0.90
	$\bar{R}$	0.0027	0.0059	0.021	0.16	0.90
	$\bar{G}$	0.0027	0.0059	0.021	0.16	0.90
	$\overline{IQR}$	0.0027	0.0057	0.020	0.15	0.88

effect of non-normality on P_δ and ARL_δ, respectively. From the tables it follows that also in this case P_0 significantly increases and so ARL_0 decreases under non-normality. The level of increase in P_0 depends on the estimator of σ. The increase in P_0 is the smallest for the $\overline{X}$ control chart based on $\tilde{S}$ and the largest for the $\overline{X}$ control chart based on $\overline{IQR}$. This is due to the fact that the estimator based on $\tilde{S}$ has a small bias under non-normality, while the bias of the estimator based on $\overline{IQR}$ is large under non-normality, see Table I. As Table I shows, this is also the case for other values of n. The performance of the other schemes is in between the performance of the charts based on $\tilde{S}$ and $\overline{IQR}$.

Corrected limits

The performance characteristics P_δ and ARL_δ of the corrected limits are presented in Tables V and VI, respectively. From Table V it follows that the differences between the charts in terms of P_δ are small. The only remarkable thing is that the charts based on $\tilde{S}$ and $\overline{IQR}$ have a slightly lower P_δ for $\delta>0$ in a number of cases. Table VI shows that the deviations between the normal and non-normal case are the smallest for the control chart based on $\bar{G}$ and therefore this chart is most robust against deviations from normality. This is due to the fact that the unbiased estimator based on $\bar{G}$ has the lowest variance in almost all cases (the variance determines the performance of the chart since the bias is removed). This can be shown by the relative efficiency of the estimators. The relative

Table VI. ARL_δ of corrected limits for $k=20$ and $n=6$

	$\hat{\sigma}$	ARL_δ for unbiased $\hat{\sigma}$				
		$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=2$
Normal	$\tilde{S}$	682	265	51.0	4.68	1.05
	$\bar{S}$	701	270	51.6	4.70	1.05
	$\bar{R}$	720	274	52.5	4.73	1.05
	$\bar{G}$	699	269	51.7	4.68	1.05
	$\overline{IQR}$	1474	509	81.3	5.11	1.07
t_4	$\tilde{S}$	769	529	321	79.3	14.7
	$\bar{S}$	503	380	179	22.5	1.27
	$\bar{R}$	511	390	184	23.2	1.27
	$\bar{G}$	484	365	169	20.4	1.23
	$\overline{IQR}$	521	391	188	23.3	1.25
t_{10}	$\tilde{S}$	717	324	76.0	6.09	1.07
	$\bar{S}$	685	309	70.5	5.97	1.07
	$\bar{R}$	701	316	71.6	6.00	1.07
	$\bar{G}$	686	309	70.4	6.00	1.07
	$\overline{IQR}$	1120	480	102	7.41	1.09
Logistic	$\tilde{S}$	735	339	77.3	6.56	1.08
	$\bar{S}$	692	322	74.0	6.38	1.07
	$\bar{R}$	738	343	77.4	6.57	1.08
	$\bar{G}$	682	318	73.3	6.34	1.07
	$\overline{IQR}$	1094	509	111	7.91	1.09
Exponential	$\tilde{S}$	3822	1703	606	68.1	2.49
	$\bar{S}$	1978	1278	432	54.9	2.22
	$\bar{R}$	2012	1279	433	54.8	2.23
	$\bar{G}$	1275	1114	380	49.0	2.11
	$\overline{IQR}$	4167	1785	601	72.3	2.54
χ^2_5	$\tilde{S}$	1276	863	232	23.3	1.37
	$\bar{S}$	1015	768	212	21.7	1.35
	$\bar{R}$	1043	775	213	21.8	1.35
	$\bar{G}$	865	713	198	20.6	1.33
	$\overline{IQR}$	2120	1179	299	28.8	1.45
χ^2_{20}	$\tilde{S}$	795	456	106	9.47	1.13
	$\bar{S}$	776	449	105	9.41	1.12
	$\bar{R}$	803	459	107	9.49	1.13
	$\bar{G}$	759	440	104	9.35	1.12
	$\overline{IQR}$	1700	758	165	12.4	1.16

efficiency of an unbiased estimator $\hat{\sigma}$ is defined as

$$\text{Reff}(\hat{\sigma}) = \frac{\text{Var}(MV)}{\text{Var}(\hat{\sigma})} * 100\%$$

where MV is the estimator out of the collection of unbiased estimators considered (in this case the estimators based on $\tilde{S}$, $\bar{S}$, $\bar{R}$, $\bar{G}$ and $\overline{IQR}$) which has minimum variance. The efficiency comparisons are presented in Table VII. This table shows that the unbiased estimator based on $\bar{G}$ has in almost all cases the lowest variance under non-normality. The unbiased estimator based on $\bar{S}$ is the second best. We can also derive from the table that the variance of the unbiased estimators based on $\tilde{S}$ and $\overline{IQR}$ is in some cases higher than the variance of the other unbiased estimators. Therefore, when the knowledge of non-normality can be used to correct the limits we recommend $\bar{G}$ instead of $\bar{S}$.

Concluding remarks

The choice of the estimator for the $\overline{X}$ control chart when the process distribution is non-normal depends on the situation at hand. When the limits based on normality are applied, the best estimator is the estimator based on $\tilde{S}$ since the resulting charts perform

Table VII. Efficiency comparisons for $k=20$

		Reff($\hat{\sigma}$) of unbiased $\hat{\sigma}$ in percentages						
n	$\hat{\sigma}$	Normal	t_4	t_{10}	Logistic	Exponential	χ^2_5	χ^2_{20}
4	$\tilde{S}$	100	48	94	94	75	84	98
	$\bar{S}$	94	89	99	99	93	95	99
	$\bar{R}$	92	90	97	97	96	97	98
	$\bar{G}$	93	95	100	100	100	100	100
	$\overline{IQR}$	86	100	96	97	100	97	94
6	$\tilde{S}$	100	43	92	92	72	80	96
	$\bar{S}$	96	86	98	98	88	91	98
	$\bar{R}$	89	81	90	89	87	89	93
	$\bar{G}$	94	100	100	100	100	100	100
	$\overline{IQR}$	48	81	57	58	68	60	53
8	$\tilde{S}$	100	42	91	91	70	79	94
	$\bar{S}$	97	80	97	96	84	88	96
	$\bar{R}$	86	70	82	82	80	82	88
	$\bar{G}$	95	100	100	100	100	100	100
	$\overline{IQR}$	56	97	68	69	78	70	62
10	$\tilde{S}$	100	39	91	91	70	78	93
	$\bar{S}$	98	76	96	95	82	86	96
	$\bar{R}$	83	61	76	76	73	76	83
	$\bar{G}$	96	100	100	100	100	100	100
	$\overline{IQR}$	43	78	53	53	64	56	48

the best both under normality and under non-normality. When the knowledge of non-normality can be used to correct the limits, the best choice is the unbiased estimator based on $\bar{G}$ since this estimator has the lowest variance under non-normality.

Note that we have performed the simulations for n varying from 4 to 10 and k equal to 20, which is in line with the assumption that in practice usually 20–30 subgroups are available of sizes around 3–10 (see e.g. Ryan[3] and Montgomery[4]). A higher value of k would moderate the effect of parameter estimation, resulting in higher probabilities of signalling in the out-of-control situation and smaller differences between the estimators.

References

1. Woodall WH, Montgomery DC. Research issues and ideas in statistical process control. *Journal of Quality Technology* 1999; **31**:376–386.
2. Quesenberry CP. The effect of sample size on estimated limits for $\bar{X}$ and X control charts. *Journal of Quality Technology* 1993; **25**:237–247.
3. Ryan TP. *Statistical Methods for Quality Improvement* (1st edn). Wiley: New York, 1989.
4. Montgomery DC. *Introduction to Statistical Quality Control* (3rd edn). Wiley: New York, 1996.
5. Schoonhoven M, Riaz M, Does RJMM. Design schemes for the $\bar{X}$ control chart. *Quality and Reliability Engineering International* 2009; DOI: 10.1002/qre.991.
6. Alwan LC, Roberts HV. The problem of misplaced control limits. *Applied Statistics* 1995; **44**:269–278.
7. Shewhart WA. *Economic Control of Quality of Manufactured Product* (1st edn). Van Nostrand Reinhold Company: New York, 1931.
8. Schilling EG, Nelson PR. The effect of non-normality on the control limits of $\bar{X}$ charts. *Journal of Quality Technology* 1976; **8**:183–188.
9. Padgett CS, Thombs LA, Padgett WJ. On the α-risks for Shewhart control charts. *Communications in Statistics—Simulation and Computation* 1992; **21**:1125–1147.
10. Burr IW. The effect of non-normality on constants for $\bar{X}$ and R charts. *Industrial Quality Control* 1967; **23**:563–568.
11. Albers W, Kallenberg WCM. Control charts in new perspective. *Sequential Analysis* 2007; **26**:123–151.
12. Duncan AJ. *Quality Control and Industrial Statistics* (4th edn). Irwin: Homewood, IL, 1974.
13. Kimball BF. On the choice of plotting positions on probability paper. *Journal of the American Statistical Association* 1960; **55**:546–560.
14. Madansky A. *Prescriptions for Working Statisticians*. Springer: New York, 1988.
15. Pearson ES, Hartley HO. *Biometrika Tables for Statisticians* (3rd edn). Cambridge University Press: Cambridge, 1970.
16. Albers W, Kallenberg WCM. Estimation in Shewhart control charts: effects and corrections. *Metrika* 2004; **59**:207–234. DOI: 10.1007/s001840300280.

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