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Research

Design Schemes for the $\bar{X}$ Control Chart

Marit Schoonhoven[1,*,†], Muhammad Riaz[2] and Ronald J. M. M. Does[3]

[1] *Institute for Business and Industrial Statistics of the University of Amsterdam (IBIS UvA), Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands*
[2] *Department of Statistics, Quaid-i-Azam University, 45320 Islamabad 44000, Pakistan*
[3] *IBIS UvA, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands*

This paper studies several issues regarding the design of the $\overline{X}$ control chart under normality. Different estimators of the standard deviation are considered and the effect of the estimator on the performance of the control chart is investigated. Furthermore, the choice of the factor used to get accurate control limits for moderate sample sizes is addressed. The paper gives an overview on the performance of the charts by studying different characteristics of the run length distribution, both in the in-control and in the out-of-control situation. Copyright © 2008 John Wiley & Sons, Ltd.

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INTRODUCTION

The $\overline{X}$ control chart is extensively used in practice to monitor the process mean. The chart consists of a graph with time on the horizontal axis, the sample mean plotted on the vertical axis and an upper and lower control limit indicating the thresholds at which the process mean can be considered out-of-control. Assuming that the process follows a normal distribution, the control limits depend on the mean μ and the standard deviation σ of the normal distribution. When both μ and σ are known the limits can be obtained directly. Let Y_{ij} $i=1,2,\ldots$ and $j=1,2,\ldots,n$ denote the jth observation in sample i. We assume that Y_{ij} are independent and $N(\mu+\delta\sigma,\sigma^2)$ distributed, where μ and σ are known and δ is a constant. When $\delta=0$, the process is in-control, otherwise the process is shifted. Then the process mean can be monitored by plotting the sample means $\bar{Y}_i=1/n\sum_{j=1}^{n}Y_{ij}$ on a traditional Shewhart control chart with limits

$$UCL=\mu+3\frac{\sigma}{\sqrt{n}},\quad LCL=\mu-3\frac{\sigma}{\sqrt{n}} \tag{1}$$

We define P_δ as the probability that a sample gives an out-of-control signal when the process mean equals $\mu+\delta\sigma$. P_δ is given by $1-\Phi(3-\delta\sqrt{n})+\Phi(-3-\delta\sqrt{n})$, where Φ denotes the standard normal distribution. Further, denote the event that the ith sample mean falls beyond the limits by E_i and the number of samples until the first sample mean falls beyond the limits when the process mean equals $\mu+\delta\sigma$ by RL_δ. As the events E_s and E_t $(s\neq t)$ are independent, RL_δ is geometrically distributed with parameter P_δ. It follows that

*Correspondence to: Marit Schoonhoven, Institute for Business and Industrial Statistics of the University of Amsterdam (IBIS UvA), Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands.
†E-mail: m.schoonhoven@uva.nl

the average run length (ARL_δ) equals $1/P_\delta$. From the preceding we can derive from (1) the probability of an incorrect signal and the average run length when the process is in-control: $P_0=0.0027$ and $ARL_0=370.4$.

In practice, μ and σ are unknown. Therefore, they must be estimated on samples taken when the process is assumed to be in-control. Woodall and Montgomery[1] define this phase as Phase I. They define the monitoring phase as Phase II. Estimating the parameters has two consequences for the performance of the control chart in Phase II. First, when the fixed constant 3 is applied for the control limits and the estimations are simply plugged into (1), P_0 will deviate from its intended value 0.0027. Second, the events E_s and E_t are no longer independent and therefore the run length distribution is not geometric. The latter issue is first addressed by Quesenberry[2]. He uses simulation to study the performance of the $\overline{X}$ control chart, for a various number of estimation samples k and sample sizes n. His results show that when the parameters are estimated on limited data, the ARL and the standard deviation of the run length increase, while the false alarm probabilities also increase. Quesenberry concludes that the number of estimation samples k should be at least $400/(n-1)$ in order to get limits that perform like known limits. This is of course unrealistic in most practical situations, where we usually have 20–30 subgroups of sizes around 3–10 (see, e.g. Ryan[3] (p. 74) and Montgomery[4] (p. 181)). This differs substantially with the recommendations of Quesenberry[2]. Nedumaran and Pignatiello[5] propose an approach for constructing control limits that attempt to match any specific percentile point of the run length distribution of the true limits (see also Albers and Kallenberg[6]). Hence, to get accurate limits for moderate sample sizes, one could consider factors that replace the fixed constant 3 in (1). Another option would be to investigate the influence of the estimator of the standard deviation in (1).

Summarizing, the issues that need to be resolved to get meaningful limits are the choice of the estimators and the determination of the factor used to get accurate limits for commonly used sample sizes. This paper studies these issues for the $\overline{X}$ control chart for various values of k and n. The mean μ in (1) is usually estimated by the mean of the sample means. The key question is the choice of the estimator of the standard deviation. Therefore, we consider different estimators of the standard deviation and investigate the effect of the estimator on control chart performance. Besides, the choice of the factor is discussed and factors different from 3 are derived. Simulations are used to study the run length characteristics of the charts, both in the in-control and in the out-of-control situation.

The paper is structured as follows. In the following section the design schemes for the $\overline{X}$ chart are presented. This section also presents the estimators that are considered and the factors that are applied. The subsequent section shows the results of the simulation of the performance of the charts, followed by the conclusions.

DESIGN SCHEMES

We shall use in this section the following manner of writing: a hat above an alphabet represents an estimator: e.g. $\hat{\mu}$. When the process parameters are unknown, the control limits become

$$\widehat{UCL}=\hat{\mu}+c(n,k,p)\frac{\hat{\sigma}}{\sqrt{n}},\quad \widehat{LCL}=\hat{\mu}-c(n,k,p)\frac{\hat{\sigma}}{\sqrt{n}} \tag{2}$$

where $c(n,k,p)$ denotes the factor that is dependent on the number of samples k, the sample size n and p, the latter being equal to P_0 (the probability that a sample gives an out-of-control signal when the process mean equals μ). Let X_{ij} $i=1,2,\ldots,k$ and $j=1,2,\ldots,n$ denote the historical data used to estimate μ and σ (Phase I) and let Y_{ij} $i=1,2,\ldots$ and $j=1,2,\ldots,n$ denote the monitoring data (Phase II). We assume that X_{ij} are $N(\mu,\sigma^2)$ distributed and Y_{ij} are $N(\mu+\delta\sigma,\sigma^2)$ distributed. When the parameters are estimated on limited data, the fixed constant 3 that is applied for the limits when the parameters are known (cf. (1)) is not adequate. Therefore, it is replaced by $c(n,k,p)$ in (2). In this section we present the estimators of μ and σ that are considered and the derivation of the factor $c(n,k,p)$.

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Estimators of location and spread

We estimate the process mean μ by the unbiased estimator

$$\bar{\bar{X}}=\frac{1}{k}\sum_{i=1}^{k}\left(\frac{1}{n}\sum_{j=1}^{n}X_{ij}\right) \tag{3}$$

i.e. the *grand sample mean*. The primary issue is the choice of the estimator of σ. We consider a number of estimators of σ. They are given below.

The first estimator of σ that we consider is based on the *pooled sample standard deviation*

$$\tilde{S}=\left(\frac{1}{k}\sum_{i=1}^{k}S_i^2\right)^{1/2} \tag{4}$$

where S_i is the ith sample standard deviation defined by

$$S_i=\left(\frac{1}{n-1}\sum_{j=1}^{n}(X_{ij}-\bar{X}_i)^2\right)^{1/2}$$

An unbiased estimator of σ is $\tilde{S}/c_4(k(n-1)+1)$, where $c_4(m)$ is defined by

$$c_4(m)=\left(\frac{2}{m-1}\right)^{1/2}\frac{\Gamma(m/2)}{\Gamma((m-1)/2)}$$

Another unbiased estimator of σ is $\bar{S}/c_4(n)$, where $\bar{S}$ is the *mean sample standard deviation*

$$\bar{S}=\frac{1}{k}\sum_{i=1}^{k}S_i \tag{5}$$

We also consider the estimator based on the *mean sample range*

$$\bar{R}=\frac{1}{k}\sum_{i=1}^{k}R_i \tag{6}$$

where R_i is the range of the ith sample. We estimate σ by the unbiased estimator $\bar{R}/d_2(n)$, where $d_2(n)$ is the expected range of a random $N(0,1)$ sample of size n. Values of $d_2(n)$ can be found in Duncan[7] (Table M).

The next estimator we propose is based on *Gini's mean sample differences*. Gini's mean differences of sample i is defined by

$$G_i=\sum_{j=1}^{n-1}\sum_{l=j+1}^{n}|X_{ij}-X_{il}|/(n(n-1)/2)$$

An unbiased estimator of σ is given by $\bar{G}/d_2(2)$, where

$$\bar{G}=\frac{1}{k}\sum_{i=1}^{k}G_i \tag{7}$$

The last estimator that we consider is based on the *mean sample interquartile range*. The interquartile range for sample i is defined by

$$IQR_i=Q_{75,i}-Q_{25,i}$$

where $Q_{r,i}$ is the rth percentile of the values in sample i. For a sample of size n, the sorted values can be interpreted as the $100(i-0.5)/n$, $i=1,2,\ldots,n$ percentiles; linear interpolation is used to compute the

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Table I. Proposed estimators of spread

$\hat{\sigma}$	
Unbiased pooled sample standard deviation	$\tilde{S}/c_4(k(n-1)+1)$
Unbiased mean sample standard deviation	$\bar{S}/c_4(n)$
Unbiased mean sample range	$\bar{R}/d_2(n)$
Unbiased Gini's mean sample differences	$\bar{G}/d_2(2)$
Unbiased mean sample interquartile range	$\overline{IQR}/q(n)$

percentiles when $100(i-0.5)/n$ is not an integer. In Kimball[8] it is shown that the best choice would be $100(i-3/8)/(n+1/4)$ instead of $100(i-0.5)/n$. We could also have used the definition that for a sample of size n, the sorted values are the $100i/n$, $i=1,2,\ldots,n$ percentiles. However, the choice $100(i-0.5)/n$ is more intuitively and better known (cf. Madansky[9]). The unbiased estimator is $\overline{IQR}/q(n)$ where

$$\overline{IQR}=\frac{1}{k}\sum_{i=1}^{k} IQR_i \tag{8}$$

and $q(n)$ is defined as the expectation of the interquartile range of a random sample of n $N(0,1)$ distributed variables. Values of $q(n)$ can be derived from the mean positions of ranked normal deviates, which are given in Table 28 in Pearson and Hartley[10].

An overview of the estimators of spread considered is given in Table I.

Derivation of the control limits

The $\hat{\mu}\pm 3\hat{\sigma}/\sqrt{n}$ limits do not perform like the known limits defined by (1) when the process parameters are estimated on limited historical data. Two issues are present. First, P_0 will deviate from the intended value 0.0027. Second, the run length distribution is no longer geometric. Quesenberry[2] argues that the number of estimation samples k should be at least $400/(n-1)$ in order to get limits that perform like known limits. However, this is not realistic for most applications.

A solution to this problem is to correct the control limits by replacing the fixed constant 3 by $c(n,k,p)$, cf. (2). However, we should keep in mind that it is not possible to construct limits that perform overall like known limits, e.g. both in terms of P_0 and ARL_0, since the run length distribution is no longer geometric. Therefore, when the limits are estimated on limited data, one should make in advance a decision on the purpose of the control chart. For example, should the chart perform well in terms of P_0, in terms of ARL_0 or in terms of a specific percentile point of the in-control run length distribution? In the literature, several suggestions for the correction of the control limits can be found. Hillier[11] determines the factors based on the values of P_0, resulting in correct false alarm probabilities. A second starting point could be to take the values of ARL_0. Taking ARL_0 as starting point is usually not the best option as ARL_0 is strongly determined by the occurrence of extreme long runs, which is often not relevant in practice, see in this respect Does and Schriever[12]. The last option we address is to determine the factor on the basis of the probability that the run length is at most a specified value x, see Nedumaran and Pignatiello[5] and Albers and Kallenberg[6].

In this study we choose to take P_0 as point of departure, i.e. we determine the factor $c(n,k,p)$ such that

$$P(\bar{Y}_i\leq L)=P\left(\sqrt{n}\frac{\bar{Y}_i-\hat{\mu}}{\hat{\sigma}}\leq c(n,k,p)\right)=p \tag{9}$$

where $L=\widehat{UCL}$ and $p=(1-0.0027)/2=0.99865$ for the upper control limit, $L=\widehat{LCL}$ and $p=0.0027/2=0.00135$ for the lower control limit, and $\bar{Y}_i$ is supposed to be in-control. In the following section we investigate the impact of this choice on the run length distribution, both in the in-control and in the out-of-control situation. Below for each estimator $\hat{\sigma}$ the factor $c(n,k,p)$ for the control chart based on $(\bar{\bar{X}},\hat{\sigma})$ is derived.

First note that all estimators $\hat{\sigma}$ of σ are stochastically independent of the estimator $\bar{\bar{X}}$ of μ, see Lehmann[13]. To derive the factor $c(n,k,p)$ for the control chart based on $\tilde{S}$, we use the fact that $(\bar{Y}_i-\bar{\bar{X}})\sqrt{nk}/(\sqrt{k+1}\tilde{S})$

Table II. Factors $c(n,k,0.99865)$ to determine control limits

n		$c(n,k,0.99865)$				
		$k=20$	$k=30$	$k=50$	$k=100$	$k=500$
4	Equation (10)	3.19	3.13	3.08	3.04	3.01
	Equation (11)	3.22	3.14	3.08	3.04	3.01
6	Equation (10)	3.14	3.10	3.06	3.03	3.01
	Equation (11)	3.17	3.11	3.07	3.04	3.01
8	Equation (10)	3.12	3.08	3.05	3.02	3.00
	Equation (11)	3.14	3.09	3.05	3.02	3.00
10	Equation (10)	3.11	3.08	3.05	3.02	3.00
	Equation (11)	3.12	3.08	3.05	3.02	3.00

has a t-distribution with $k(n-1)$ degrees of freedom. It follows that the factor $c(n,k,p)$ is defined by

$$c(n,k,p)=c_4(k(n-1)+1)\sqrt{k+1}\,t_{k(n-1)}(p)/\sqrt{k} \tag{10}$$

where $t_{k(n-1)}(p)$ denotes the pth percentile of a t-distribution with $k(n-1)$ degrees of freedom.

The factors $c(n,k,p)$ for the control charts based on the estimators defined by (5)–(8) are not derived exactly since the distribution of $(\bar{Y}_i-\bar{\bar{X}})/\hat{\sigma}$ is hard to find. We consider the following approximations for $c(n,k,p)$. First we approximate the distribution of the unbiased estimators based on (5)–(8) simply by the distribution of $\tilde{S}/c_4(k(n-1)+1)$. As a result, the approximation of $c(n,k,p)$ is given by (10).

For the factor $c(n,k,p)$, for the control chart based on $\bar{R}$, we also consider an alternative approximation based on a result of Patnaik[14]. He approximates the distribution of $\bar{R}/\sigma$ by $a(n,k)\chi_{v(n,k)}/\sqrt{v(n,k)}$, where $\chi_{v(n,k)}$ is the square root of a chi-square distribution with $v(n,k)$ degrees of freedom and $a(n,k)$ is a scale factor. The factors $a(n,k)$ and $v(n,k)$ are obtained by equating the first two moments of $\bar{R}/\sigma$ to the first two moments of $a(n,k)\chi_{v(n,k)}/\sqrt{v(n,k)}$. These values are given in Table 7.3.2 of David[15]. Hence, $(\bar{Y}_i-\bar{\bar{X}})/\hat{\sigma}$ is approximately distributed as $d_2(n)\sqrt{k+1}\,t_{v(n,k)}(p)/(a(n,k)\sqrt{k})$. The approximation of $c(n,k,p)$ becomes

$$c(n,k,p)\cong d_2(n)\sqrt{k+1}\,t_{v(n,k)}(p)/(a(n,k)\sqrt{k}) \tag{11}$$

Values of $c(n,k,0.99865)$ obtained by (10) and (11) are given in Table II, for various values of n and k. The table shows that the values obtained by (10) are close to the values obtained by the regularly applied but more complex equation (11). Therefore, from now on we will use (10) to obtain $c(n,k,p)$ for all estimators considered.

EVALUATION

In this section the design schemes presented in the previous section are evaluated. The performance of the schemes is measured in terms of the probability of a signal in an individual sample (P_δ), the average run length (ARL_δ) and the run length distribution ($P(RL_\delta\leq x)$), for δ equal to 0, 0.25, 0.5, 1 and 2. We use simulation to obtain these performance characteristics. The simulation procedure is described in the first paragraph of the section. The second and third paragraphs present the simulation results for the in-control situation ($\delta=0$) and the out-of-control situation ($\delta=0.25,0.5,1,2$), respectively.

Simulation procedure

Let X_{ij} $i=1,2,\ldots,k$ and $j=1,2,\ldots,n$ denote the data collected in Phase I to estimate the process parameters. X_{ij} is assumed to be statistically in-control. Let Y_{ij} $i=1,2,\ldots$ and $j=1,2,\ldots,n$ denote the monitoring data and denote the conditional probability that for given $\bar{\bar{X}}$ and $\hat{\sigma}$ by $P(E_i|\bar{\bar{X}},\hat{\sigma})$, the sample mean $\bar{Y}_i$ falls

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beyond the control limits

$$P(E_i|\bar{\bar{X}},\hat{\sigma}) = P(\bar{Y}_i < \widehat{LCL} \text{ or } \bar{Y}_i > \widehat{UCL})$$

Given $\bar{\bar{X}}$ and $\hat{\sigma}$, the events E_s and E_t $(s \neq t)$ are independent. Therefore, the run length RL has a geometric distribution with parameter $P(E_i|\bar{\bar{X}},\widehat{\sigma})$

$$P(RL \leq x|\bar{\bar{X}},\hat{\sigma}) = 1-(1-P(E_i|\bar{\bar{X}},\widehat{\sigma}))^x$$

and an average run length

$$E(RL|\bar{\bar{X}},\hat{\sigma}) = 1/P(E_i|\bar{\bar{X}},\widehat{\sigma})$$

When we take the expectation over the estimation data X_{ij} we get the unconditional probability of one sample showing a false alarm

$$P(RL=1) = E(P(E_i|\bar{\bar{X}},\widehat{\sigma}))$$

the unconditional run length distribution

$$P(RL \leq x) = E(1-(1-P(E_i|\bar{\bar{X}},\widehat{\sigma}))^x)$$

and the unconditional ARL

$$ARL = E(1/P(E_i|\bar{\bar{X}},\widehat{\sigma}))$$

These expectations have been obtained by simulation. We have generated 100 000 times k data samples of size n and for each data set we have computed the conditional value of each performance characteristic. The unconditional values have been obtained by averaging the conditional values over the data sets. We have performed these simulations for a number of samples k equal to 20, 30, 50, 100 and 500, and sample sizes n equal to 4, 6, 8 and 10.

The in-control situation

In this section we study the performance of the design schemes in the in-control situation. First, we study the performance in terms of the probability of a false alarm (P_0). Values of P_0 are presented in Table III. As the factors $c(n,k,p)$ have been derived by controlling P_0, we do not expect that P_0 differs much from its intended value 0.0027. For the control chart based on $\tilde{S}$, $c(n,k,p)$ has been derived exactly, thus, P_0 is exactly 0.0027 in this case, while when the traditional $3\hat{\sigma}$ limits had been used P_0 would have been significantly higher: e.g. for $n=4$, 0.0047 for $k=20$ and 0.0034 for $k=50$. For the charts based on the other estimators of σ (cf. (5)–(8)), the factors have been approximated. As can be derived from Table III, these approximations are accurate except for the control chart based on $\overline{IQR}$ when n is equal to 6, 8 or 10. This is due to the fact that $c(n,k,p)$ has been obtained by approximating the distribution of $\hat{\sigma}$ by the distribution of $\tilde{S}/c_4(k(n-1)+1)$, while the distribution of $\overline{IQR}/q(n)$ has heavier tails for these values of n. This can be shown by the relative efficiency of the estimators, i.e. the variance of the estimators compared with the variance of the estimator $\tilde{S}/c_4(k(n-1)+1)$, which is an unbiased estimator with minimum variance. The relative efficiency is defined by

$$\text{Reff}(\hat{\sigma}) = \frac{\text{Var}(\tilde{S}/c_4(k(n-1)+1))}{\text{Var}(\hat{\sigma})} * 100\%$$

The efficiency comparisons are presented in Table IV. The table shows that the variance of $\overline{IQR}/q(n)$ is considerably higher than the variance of the other estimators considered, especially for n equal to 6, 8 and 10. This results in more variable control limits. In order to make a fair comparison between the estimators in the sequel of the study, we fix $P_0 = 0.0027$ for the chart based on $\overline{IQR}$ and recalculate the factor $c(n,k,p)$. The resulting factors are presented in Table V.

Table III. P_0 of proposed charts (2) and traditional $3\hat{\sigma}$ charts

		$P_0 \times 10^2$ for unbiased $\hat{\sigma}$						
		Proposed charts					Traditional charts	
n	$\hat{\sigma}$	$k=20$	$k=30$	$k=50$	$k=100$	$k=500$	$k=20$	$k=50$
4	$\tilde{S}$	0.27	0.27	0.27	0.27	0.27	0.47	0.34
	$\bar{S}$	0.28	0.27	0.27	0.27	0.27	0.47	0.34
	$\bar{R}$	0.28	0.28	0.27	0.27	0.27	0.48	0.35
	$\bar{G}$	0.28	0.28	0.27	0.27	0.27	0.48	0.34
	$\overline{IQR}$	0.28	0.28	0.28	0.27	0.27	0.48	0.35
6	$\tilde{S}$	0.27	0.27	0.27	0.27	0.27	0.41	0.32
	$\bar{S}$	0.27	0.27	0.27	0.27	0.27	0.42	0.32
	$\bar{R}$	0.28	0.27	0.27	0.27	0.27	0.42	0.33
	$\bar{G}$	0.27	0.27	0.27	0.27	0.27	0.42	0.33
	$\overline{IQR}$	0.34	0.32	0.30	0.28	0.27	0.50	0.35
8	$\tilde{S}$	0.27	0.27	0.27	0.27	0.27	0.39	0.31
	$\bar{S}$	0.27	0.27	0.27	0.27	0.27	0.39	0.32
	$\bar{R}$	0.28	0.27	0.27	0.27	0.27	0.40	0.32
	$\bar{G}$	0.27	0.27	0.27	0.27	0.27	0.40	0.32
	$\overline{IQR}$	0.30	0.29	0.28	0.28	0.27	0.43	0.33
10	$\tilde{S}$	0.27	0.27	0.27	0.27	0.27	0.38	0.31
	$\bar{S}$	0.27	0.27	0.27	0.27	0.27	0.38	0.31
	$\bar{R}$	0.28	0.27	0.27	0.27	0.27	0.39	0.31
	$\bar{G}$	0.27	0.27	0.27	0.27	0.27	0.38	0.31
	$\overline{IQR}$	0.31	0.30	0.29	0.28	0.27	0.44	0.33

Table IV. Efficiency comparison

	Reff ($\hat{\sigma}$) of unbiased $\hat{\sigma}$ in percentages			
$\hat{\sigma}$	$n=4$	$n=6$	$n=8$	$n=10$
$\tilde{S}$	100	100	100	100
$\bar{S}$	94	96	97	98
$\bar{R}$	92	89	86	83
$\bar{G}$	93	94	95	96
$\overline{IQR}$	86	48	56	43

Table V. Factors $c(n, k, 0.99865)$ for control charts when σ is estimated by $\overline{IQR}/q(n)$

	$c(n, k, 0.99865)$				
n	$k=20$	$k=30$	$k=50$	$k=100$	$k=500$
4	3.19	3.13	3.08	3.04	3.01
6	3.23	3.15	3.09	3.05	3.01
8	3.16	3.11	3.06	3.03	3.00
10	3.16	3.11	3.06	3.03	3.00

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Table VI. ARL_0 of control charts based on k samples of size n

		ARL_0 for unbiased $\hat{\sigma}$					
n	$\hat{\sigma}$	$k=20$	$k=30$	$k=50$	$k=100$	$k=500$	$k=\infty$
4	$\tilde{S}$	1069	702	532	439	383	370
	$\bar{S}$	1110	725	540	442	383	
	$\bar{R}$	1145	737	541	443	383	
	$\bar{G}$	1141	733	540	443	384	
	$\overline{IQR}$	1207	762	554	449	385	
6	$\tilde{S}$	682	545	461	411	378	
	$\bar{S}$	701	547	462	412	378	
	$\bar{R}$	720	561	469	415	379	
	$\bar{G}$	699	551	464	412	378	
	$\overline{IQR}$	1474	848	589	453	381	
8	$\tilde{S}$	580	489	433	399	375	
	$\bar{S}$	584	492	434	400	375	
	$\bar{R}$	603	504	439	402	377	
	$\bar{G}$	588	493	435	400	376	
	$\overline{IQR}$	805	608	479	419	378	
10	$\tilde{S}$	530	462	421	393	374	
	$\bar{S}$	532	465	421	392	375	
	$\bar{R}$	549	474	424	394	374	
	$\bar{G}$	534	465	421	393	374	
	$\overline{IQR}$	812	613	482	419	378	

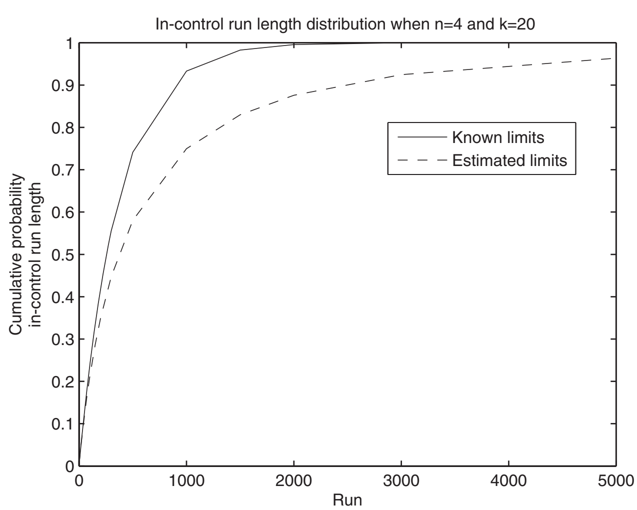

Figure 1. In-control run length distribution when $n=4$ and $k=20$

A higher variance also causes that the ARL_0 would be higher, as there will be more extreme runs. Values of ARL_0 are presented in Table VI. It is remarkable that the control chart based on $\overline{IQR}$ has a higher ARL_0 for $n=6$ compared with $n=4$, which is also the case for $n=10$ compared with $n=8$. This is due to the fact that the variance of the estimator is higher for these values of n. Riaz[16] studies extensively the control

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chart based on the interquartile range. He also notes that for different values of n the design structure of the control chart based on the interquartile range shows irregular patters (see also Table I of Rocke[17]).

In general, from Table VI it turns out that the ARL_0 is significantly higher than the intended 370 when we fix $P_0 = 0.0027$. However, the relevance of the ARL is questioned as its value is strongly determined by the occurrence of extreme runs, while in practice processes do not remain unchanged for a very long period. A more relevant performance measure is the run length distribution.

Figures 1 and 2 show the in-control run length distribution for $n=4$ and 6, respectively, and $k=20$. It turns out that the charts based on the different estimators perform similarly for $n=4$, but for $n=6$ the control chart based on $\overline{IQR}$ loses power. In general, the choice of constructing the limits by controlling P_0 seems to be suitable also for short monitoring runs (e.g. start-ups). When the monitoring period is expected

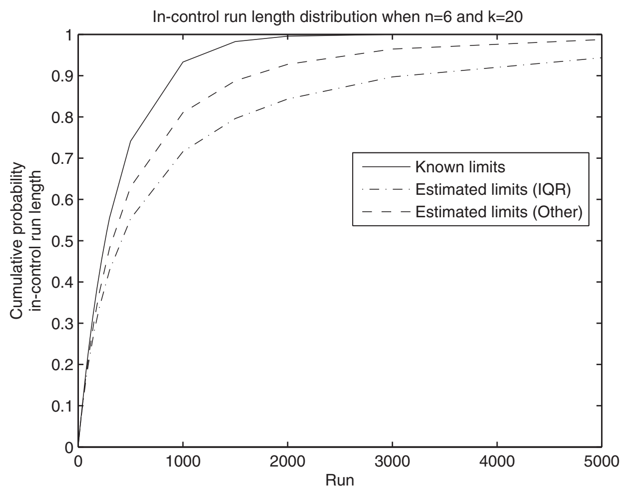

Figure 2. In-control run length distribution when $n=6$ and $k=20$

Table VII. P_δ of control charts based on k samples of size n

			P_δ for unbiased $\hat{\sigma}$					
	n	$\hat{\sigma}$	$k=20$	$k=30$	$k=50$	$k=100$	$k=500$	$k=\infty$
$P_{0.25}\times 10^2$	4	$\tilde{S}$	0.59	0.61	0.62	0.64	0.64	0.64
		$\bar{S}$	0.60	0.62	0.63	0.64	0.64	0.64
		$\bar{R}, \bar{G}$	0.61	0.62	0.63	0.64	0.64	0.64
		$\overline{IQR}$	0.62	0.63	0.63	0.64	0.64	0.64
	6	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}$	0.81	0.83	0.84	0.85	0.86	0.86
		$\overline{IQR}$	0.77	0.80	0.82	0.85	0.86	0.86
$P_{0.5}\times 10$	4	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}, I\bar{Q}R$	0.20	0.21	0.22	0.22	0.23	0.23
	6	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}$	0.34	0.35	0.36	0.37	0.38	0.38
		$\overline{IQR}$	0.32	0.33	0.35	0.36	0.38	0.38
P_1	4	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}, I\bar{Q}R$	0.13	0.14	0.15	0.15	0.16	0.16
	6	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}$	0.25	0.27	0.28	0.28	0.29	0.29
		$\overline{IQR}$	0.24	0.25	0.27	0.28	0.29	0.29
P_2	4	$\tilde{S}$	0.78	0.80	0.82	0.83	0.84	0.84
		$\bar{S}, \bar{R}, \bar{G}, \overline{IQR}$	0.77	0.80	0.82	0.83	0.84	0.84
	6	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}$	0.95	0.96	0.96	0.97	0.97	0.97
		$\overline{IQR}$	0.94	0.95	0.96	0.97	0.97	0.97

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Table VIII. ARL_δ of control charts based on k samples of size n

			ARL_δ for unbiased $\hat{\sigma}$					
δ	n	$\hat{\sigma}$	$k=20$	$k=30$	$k=50$	$k=100$	$k=500$	$k=\infty$
0.25	4	$\tilde{S}$	528	331	241	192	162	155
		$\bar{S}$	547	342	243	193	162	
		$\bar{R}$	554	344	244	193	162	
		$\bar{G}$	553	343	244	192	162	
		$\overline{IQR}$	590	354	249	195	163	
	6	$\tilde{S}$	265	198	159	135	119	116
		$\bar{S}$	270	199	159	135	119	
		$\bar{R}$	274	201	161	136	120	
		$\bar{G}$	269	200	160	135	120	
		$\overline{IQR}$	509	286	193	148	120	
0.5	4	$\tilde{S}$	127	83.9	65.6	52.4	45.4	43.9
		$\bar{S}$	129	85.6	64.1	52.6	45.5	
		$\bar{R}$	132	85.3	64.0	52.5	45.5	
		$\bar{G}$	131	85.7	63.9	52.6	45.5	
		$\overline{IQR}$	136	87.7	65.1	52.9	45.6	
	6	$\tilde{S}$	51.0	40.0	33.6	29.7	27.0	26.4
		$\bar{S}$	51.6	40.4	33.7	29.6	27.0	
		$\bar{R}$	52.5	40.7	33.9	29.8	27.0	
		$\bar{G}$	51.7	40.3	33.7	29.7	27.0	
		$\overline{IQR}$	81.3	52.3	38.8	31.7	27.0	
1	4	$\tilde{S}$	11.4	9.08	7.76	6.97	6.43	6.30
		$\bar{S}$	11.6	9.14	7.79	6.97	6.43	
		$\bar{R}$	11.6	9.17	7.80	6.98	6.43	
		$\bar{G}$	11.6	9.18	7.80	6.98	6.43	
		$\overline{IQR}$	11.8	9.27	7.84	7.01	6.44	
	6	$\tilde{S}$	4.68	4.18	3.85	3.63	3.47	3.44
		$\bar{S}$	4.70	4.20	3.85	3.63	3.47	
		$\bar{R}$	4.73	4.20	3.86	3.64	3.48	
		$\bar{G}$	4.68	4.19	3.86	3.64	3.47	
		$\overline{IQR}$	5.11	4.75	3.94	3.74	3.47	
2	4	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}$	1.32	1.27	1.23	1.21	1.19	1.19
		$\overline{IQR}$	1.33	1.27	1.23	1.21	1.19	
	6	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}$	1.05	1.04	1.04	1.03	1.03	1.03
		$\overline{IQR}$	1.07	1.05	1.04	1.04	1.03	

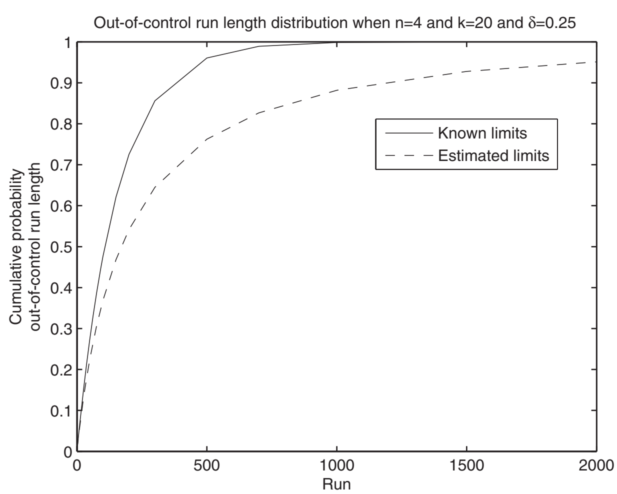

Figure 3. Out-of-control run length distribution when $n=4$ and $k=20$ and delta$=0.25$

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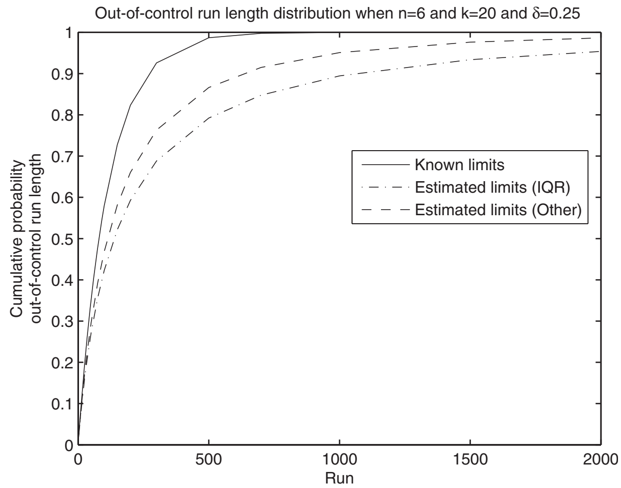

Figure 4. Out-of-control run length distribution when $n=6$ and $k=20$ and delta$=0.25$

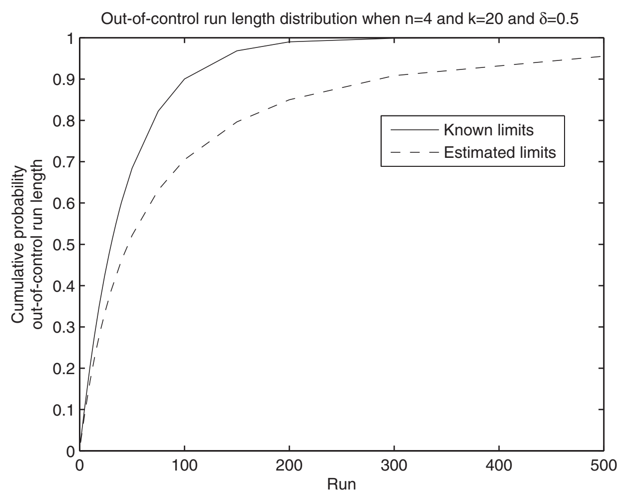

Figure 5. Out-of-control run length distribution when $n=4$ and $k=20$ and delta$=0.5$

to be longer, we recommend other factors, for example, the one that determines the factor based on the probability that the run length is at most a specified value x, see Albers and Kallenberg[6].

The out-of-control situation

In this section we investigate the performance of the $\overline{X}$ design schemes when the process is out-of-control. The out-of-control situations that we consider concern shifts in the mean to a level of $\mu+\delta\sigma$, for δ equal to 0.25, 0.5, 1 and 2. We study the same performance characteristics as in the in-control situation: P_δ (Table VII), ARL_δ (Table VIII) and $P(RL_\delta \leq x)$ (Figures 3–8 and Tables IX and X). Table VII indicates that the probabilities P_δ of the charts based on (4)–(7) are acceptable. This means that these charts are suitable for the purpose for which they have been derived, that is, approaching the intended value of P_δ. The chart based on $\overline{IQR}$ turns out to be less powerful in the out-of-control situation for n equal to 6 as the P_δ is lower, ARL_δ higher and the run length distribution is less powerful compared with the other estimators. This

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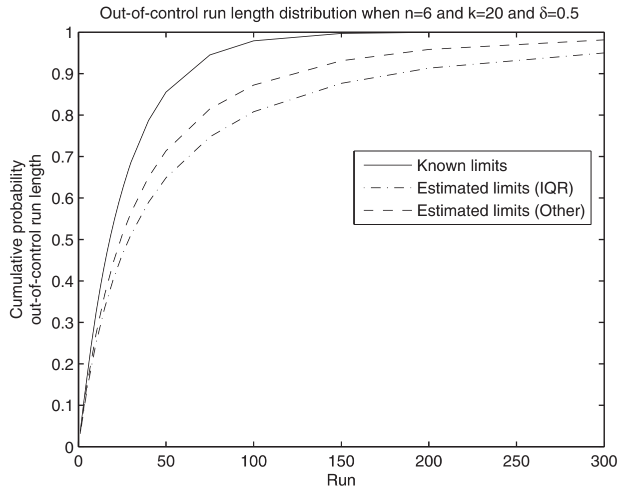

Figure 6. Out-of-control run length distribution when $n=6$ and $k=20$ and delta$=0.5$

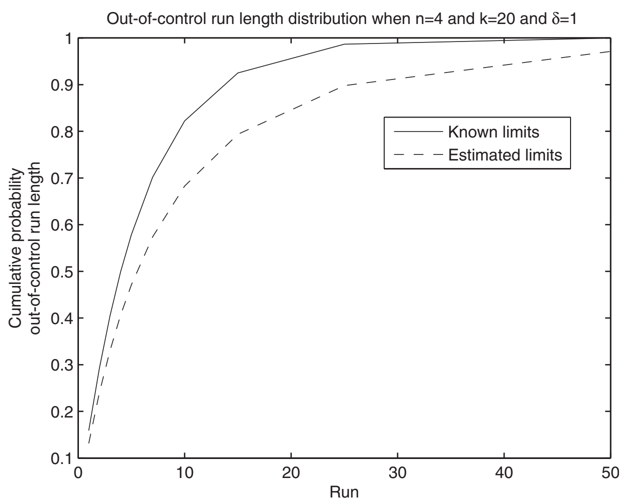

Figure 7. Out-of-control run length distribution when $n=4$ and $k=20$ and delta$=1$

is also the case for $n=8$ and 10. This is caused by the higher limits of the $\overline{IQR}$ chart that are needed to control P_0.

CONCLUDING REMARKS

In this paper two issues relevant to the design of the $\overline{X}$ chart have been studied. First, the impact of the estimator on the control chart performance is investigated. Different estimators are considered. When the estimator has a higher variance, in case of $\overline{IQR}/q(n)$ when n is equal to 6, 8 and 10, the tails of the run length distribution are heavier. In our case, the charts have been constructed by fixing the false alarm probability (P_0); hence, the right tail of the run length distribution is heavier, i.e. there are more extreme runs resulting also in a higher ARL_0. Of course, when the charts had been constructed by fixing ARL_0, the opposite would

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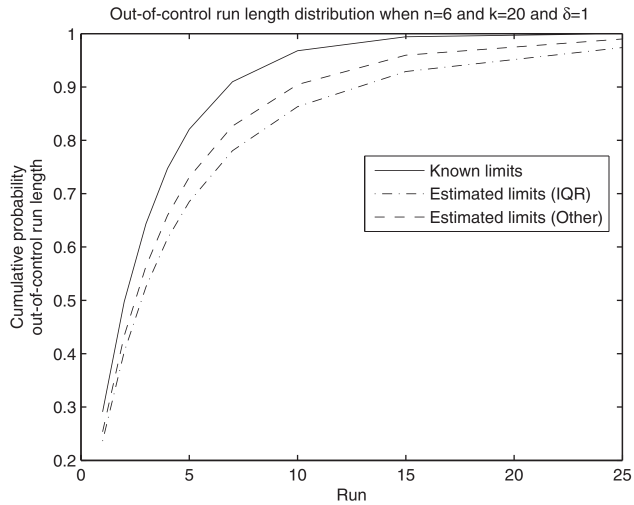

Figure 8. Out-of-control run length distribution when $n=6$ and $k=20$ and delta$=1$

Table IX. Out-of-control run length distribution when $n=4$ and $k=20$ and $\delta=2$

	$P(RL_2 \le x)$		
x	Estimated limits		Known limits
1	$\tilde{S}$	0.78	0.84
	$\bar{S}, \bar{R}, \bar{G}, \overline{IQR}$	0.77	0.84
2	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}, \overline{IQR}$	0.94	0.97
3	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}, \overline{IQR}$	0.98	1.00
4	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}, \overline{IQR}$	0.99	1.00
5	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}, \overline{IQR}$	1.00	1.00

Table X. Out-of-control run length distribution when $n=6$ and $k=20$ and $\delta=2$

	$P(RL_2 \le x)$		
x	Estimated limits		Known limits
1	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}$	0.95	0.97
	$\overline{IQR}$	0.94	0.97
2	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}$	1.00	1.00
	$\overline{IQR}$	0.99	1.00
3	$\tilde{S}, \bar{S}, \bar{R}, \bar{G}, \overline{IQR}$	1.00	1.00

have been the case. In addition, in the out-of-control situation the $\overline{IQR}$ chart shows less power in terms of P_δ and ARL_δ. The differences between the performance of the charts based on $\tilde{S}$, $\bar{S}$, $\bar{R}$ and $\bar{G}$ are smaller. These charts all perform well for the purpose for which they have been constructed, that is, approaching the intended value of P_δ.

The second issue that has been addressed is the choice of the factor $c(n,k,p)$ that replaces the fixed constant 3 in the traditional limits, in order to obtain accurate limits for moderate sample sizes. We have

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explained that it depends on the purpose of the control chart which factor suits the best. For example, the factor $c(n,k,p)$ can be constructed by fixing the in-control probability of a signal in one sample or by fixing another specific point of the in-control run length distribution such that the chart performs well for several runs. We have chosen to construct the limits by fixing P_0 and showed the impact of this choice on the run length distribution. The resulting factors can be obtained more easily compared to the factors based on the Patnaik approximation and they have shown to perform well for the estimators (4)–(7) in terms of the probability of signalling. In addition, for short monitoring runs they perform well. When longer monitoring runs are considered, we suggest to derive $c(n,k,p)$ on the basis of a specific point of the in-control run length distribution.

If we also evaluate the smaller differences, then it becomes clear that the best choice is the control chart based on $\tilde{S}$ as was to be expected. The differences between the control charts based on $\bar{S}$ and $\bar{G}$ are negligible. The performance of the control chart based on $\bar{R}$ is slightly less than the ones based on $\bar{S}$ and $\bar{G}$, mainly for higher values of n.

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Authors' biographies

Marit Schoonhoven is a consultant at the Institute of Business and Industrial Statistics of the University of Amsterdam (IBIS UvA). She is working on a PhD project focussing on control charting techniques. Her e-mail address is m.schoonhoven@uva.nl.

Muhammad Riaz is an Assistant Professor at the Department of Statistics of the Quad-i-Azam University. His e-mail address is riaz76qau@yahoo.com.

Ronald J. M. M. Does is a Professor in Industrial Statistics at the Department of Mathematics of the University of Amsterdam and Managing Director of IBIS UvA. His e-mail address is r.j.m.m.does@uva.nl.

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