Quality Engineering
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597292

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Online Publication Date: 01 January 2009

To cite this Article Riaz, Muhammad and Does, Ronald J. M. M.(2009)'An Alternative to the Bivariate Control Chart for Process Dispersion',Quality Engineering,21:1,63 — 71

To link to this Article: DOI: 10.1080/08982110802445579

URL: http://dx.doi.org/10.1080/08982110802445579

Quality Engineering, 21:63–71, 2009

ISSN: 0898-2112 print/1532-4222 online
DOI: 10.1080/08982110802445579

Taylor & Francis
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An Alternative to the Bivariate Control Chart for Process Dispersion

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ABSTRACT In this article, we propose a Shewhart-type control chart for monitoring changes in the process variability of a bivariate process. The sample Gini mean differences based matrix $|\underset{\sim}{G}|$ is used as an estimate of the population variance–covariance matrix $\underset{\sim}{\Sigma}$. The newly proposed control chart, denoted by $|\underset{\sim}{G}|$-chart, is based on the generalized Gini mean differences $|\underset{\sim}{G}|$. For the case of two correlated quality characteristics Y and X, the design structure of the proposed $|\underset{\sim}{G}|$-chart is developed assuming bivariate normality of (Y, X). The performance of the proposed $|\underset{\sim}{G}|$-chart is compared with that of the $|\underset{\sim}{S}|$-chart (a sample generalized variance based control chart).

KEYWORDS average run length, control charts, generalized variance, non-normality, normality, process variability

INTRODUCTION

Multivariate statistical process control (MSPC) is used to simultaneously monitor multiple measurements from a process. Multiple process variables might be measured such as temperatures, pressures, concentrations, flow rates, voltages, et cetera. With multiple measurements, each can be monitored in its own control chart. However, this has two disadvantages (see Runger, 2007). One is that it is difficult to control the number of false alarms. The other is that there are often important relations between the variables that should be considered for MSPC. In a multivariate setup, the variance–covariance matrix $\underset{\sim}{\Sigma}$ and the mean vector $\underline{\mu}$ are generally used to refer to the spread and location parameters respectively of the distribution of a random vector $\underline{X}$. Note that we shall use the following manner of writing: an italic character represents a univariate quantity; e.g., Y; an underlined character represents a vector; e.g., $\underline{\mu}$; a tilde under a character represents a matrix; e.g., $\underset{\sim}{\Sigma}$; $|\underset{\sim}{\Sigma}|$ represents the determinant of the matrix $\underset{\sim}{\Sigma}$; and a hat above a character represents an estimate; e.g., $|\hat{\underset{\sim}{\Sigma}}|^{1/2}$.

Several papers are available in the quality control literature that provide an extensive review of multivariate control charts (cf. Bersimis et al., 2007; Lowry and Montgomery, 1995; Wierda, 1994; Yeh et al., 2006).

Multivariate control charts for controlling the process mean were introduced by Hotelling (1947). Process variability is summarized in the

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variance–covariance matrix $\underset{\sim}{\Sigma}$. There are two single-number quantities for measuring the overall variability of a set of multivariate data. These are (1) the determinant of the variance–covariance matrix, $|\underset{\sim}{\Sigma}|$, which is called the generalized variance; (2) the trace of the variance–covariance matrix, $\mathrm{tr}\underset{\sim}{\Sigma}$, which is the sum of the variances of the variables.

The issue of multivariate monitoring of process variability is addressed by different researchers (cf. Alt, 1985; Alt and Smith, 1988; Aparaisi et al., 1999, 2001; Khoo and Quah, 2004; Menzyfricke, 2007; Montgomery and Wadsworth, 1972). Alt (1985) and Alt and Smith (1988) gave different approaches for monitoring variability of normally distributed quality characteristics in a process, of which the sample generalized variance based control chart (i.e., the $|\underset{\sim}{S}|$-chart) is the one that is commonly used. The $|\underset{\sim}{S}|$-chart is the standard multivariate control chart for dispersion in statistical software programs, like Minitab 15. In this article we shall also use the generalized variance as a measure of the process dispersion. Although the generalized variance is a widely used measure of variability, it can be misleading in some cases (cf. Lowry and Montgomery, 1995). The reason is that the values of the generalized variance do not represent unique correlations for the underlying variables.

The focus of this article is on monitoring the variability parameter with respect to two correlated quality characteristics having a bivariate normal distribution, following Alt (1985), Alt and Smith (1988), Aparaisi et al. (1999), and Khoo and Quah (2004). They have given a relationship between $|\underset{\sim}{S}|^{1/2}$ and $|\underset{\sim}{\Sigma}|^{1/2}$ to develop the design structure of the $|\underset{\sim}{S}|$ control chart, where $|\underset{\sim}{S}|^{1/2}$ and $|\underset{\sim}{\Sigma}|^{1/2}$ are the square roots of the determinants. Under bivariate normality $A = 2(n-1)|\underset{\sim}{S}|^{1/2}/|\underset{\sim}{\Sigma}|^{1/2}$ has a chi-square distribution with $2n-4$ degrees of freedom (cf. Khoo and Quah, 2004). Based on this property the probability and the 3-sigma limits based design structure for the $|\underset{\sim}{S}|$ -chart is easily obtained.

The $|\underset{\sim}{S}|$-chart performs well when the vector $\underline{X}$, of the correlated quality characteristics of interest, follows a multivariate normal distribution. In cases of contaminated multivariate normal distributions and departures from multivariate normality, the $|\underset{\sim}{S}|$-chart loses its efficiency. To overcome these problems with the $|\underset{\sim}{S}|$-chart, this article proposes a Shewhart-type control chart based on the sample Gini mean differences based matrix, say $\underset{\sim}{G}$, as an estimate of the population variance-covariance matrix $\underset{\sim}{\Sigma}$. The generalized Gini mean differences, say $|\underset{\sim}{G}|$, based control chart, denoted by $|\underset{\sim}{G}|$-chart, is proposed to monitor the population-generalized variance $|\underset{\sim}{\Sigma}|$. The motivation for this is to obtain control limits that are more robust so that these are less affected by departures from normality. Note that Riaz and Saghir (2007) proposed a Gini mean differences based univariate control chart for monitoring the scale parameter of a normally distributed quality characteristic.

One may object that multivariate methods have not gained much popularity on the shop floor. This is probably due to an important drawback: the interpretation of out-of-control situations signaled by a multivariate chart is usually difficult and involves further statistical evaluation of the data. However, in chapters 7 and 8 in Mason and Young (2002), one finds guidelines on interpretation multivariate control charts and interpretation after a signal. Does et al. (1999) Woodall and Ncube (1985) and demonstrate in their papers that simultaneous univariate charts often perform as well as multivariate charts. Note that we can always use univariate control charts for variability as a supplement to any control chart based on the generalized variance.

This article contains the following topics:

1. The design structure of the proposed $|\underset{\sim}{G}|$-chart is developed for the bivariate case assuming bivariate normality. A comparison of the $|\underset{\sim}{G}|$-chart is made with the $|\underset{\sim}{S}|$-chart in terms of average run length (ARL).
2. The robustness against departures from bivariate normality is examined on the design structures of the $|\underset{\sim}{G}|$- and $|\underset{\sim}{S}|$-charts. To examine the robustness of the design structures of the $|\underset{\sim}{G}|$- and $|\underset{\sim}{S}|$-charts, the affected ARLs (i.e., when the parent distribution is either bivariate t, bivariate chi square or bivariate exponential) have been compared with the respective original ARLs (i.e., when the parent distribution is bivariate normal).

THE PROPOSED CONTROL CHART ($|\underset{\sim}{G}|$-CHART)

Let Y and X be two correlated quality characteristics that follow a bivariate normal distribution; i.e., $(Y,X) \sim N_2(\underline{\mu}, \underset{\sim}{\Sigma})$ where $\underline{\mu} = \begin{pmatrix} \mu_y \\ \mu_x \end{pmatrix}$ and

$\underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_y^2 & \sigma_{yx} \\ \sigma_{xy} & \sigma_x^2 \end{bmatrix}$ (symbolically, the same may also be written in another way as: $(Y,X) \sim N_2(\mu_y, \mu_x, \sigma_y^2, \sigma_x^2, \rho_{yx})$). The most commonly used measure of multivariate process variability is the variance-covariance matrix $\underset{\sim}{\Sigma}$. A single number representation for the variation expressed by the matrix $\underset{\sim}{\Sigma}$ is its determinant, known as the generalized variance. The population generalized variance is denoted by $|\underset{\sim}{\Sigma}|$ and our interest in this study lies in monitoring $|\underset{\sim}{\Sigma}|^{1/2}$.

Let $(y_1, x_1), (y_2, x_2), \ldots, (y_n, x_n)$ be a random sample of size n from $N_2(\underline{\mu}, \underset{\sim}{\Sigma})$, then the sample Gini mean differences based matrix $\underset{\sim}{G}$ is defined as (cf. David, 1968; Olkin and Yitzhaki, 1992; Riaz and Saghir, 2007; Schechtman and Yitzhaki, 1987; Yitzhaki, 2003):

$$\underset{\sim}{G} = \begin{bmatrix} G_y^2 & G_{yx} \\ G_{xy} & G_x^2 \end{bmatrix},$$

$$\text{with} \quad \left.\begin{aligned} G_y &= (\sqrt{\pi}/2)4Cov(Y, F(Y)) \\ G_x &= (\sqrt{\pi}/2)4Cov(X, F(X)) \\ G_{yx} &= (\sqrt{\pi}/2)4Cov(Y, F(X))G_x \\ G_{xy} &= (\sqrt{\pi}/2)4Cov(X, F(Y))G_y \end{aligned}\right\}, \quad [1]$$

where the symbols F and Cov represent the cumulative distribution function and the covariance, respectively. The sample versions of all the quantities in the formulas of G_y, G_x, G_{yx}, and G_{xy} are used here. The generalized Gini mean differences, denoted by $|\underset{\sim}{G}|$, are defined as $|\underset{\sim}{G}| = G_y^2 G_x^2 - G_{yx} G_{xy}$. The elements of the matrix $\underset{\sim}{G}$ estimate the respective elements of the variance-covariance matrix $\underset{\sim}{\Sigma}$, and hence the quantity $|\underset{\sim}{G}|$ is used as an estimate of the $|\underset{\sim}{\Sigma}|$. In this study, the quantity $|\underset{\sim}{G}|^{1/2}$ is used for monitoring the quantity $|\underset{\sim}{\Sigma}|^{1/2}$ and for developing the design structure of the proposed $|\underset{\sim}{G}|$-chart. The quantity $|\underset{\sim}{G}|^{1/2}$ represents the square root of the determinant of the Gini's mean differences based matrix $\underset{\sim}{G}$.

To develop the design structure of the proposed $|\underset{\sim}{G}|$-chart, let B be a random variable that defines a relationship between $|\underset{\sim}{G}|^{1/2}$ and $|\underset{\sim}{\Sigma}|^{1/2}$ as:

$$B = 2(n-1)|\underset{\sim}{G}|^{1/2} / |\underset{\sim}{\Sigma}|^{1/2}, \quad [2]$$

The distributional behavior of B (in terms of its mean, standard error, and quantile points) is required for the development of the design structure of the proposed $|\underset{\sim}{G}|$-chart.

Some Distributional Results for the Proposed $|\underset{\sim}{G}|$-Chart

Assuming $(Y,X) \sim N_2(\underline{\mu}, \underset{\sim}{\Sigma})$, we consider here, without loss of generality, a standard bivariate normal distribution (i.e., $(Y,X) \sim N_2(\underline{0}, \underset{\sim}{\rho})$ where $\underline{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\underline{\rho} = \begin{bmatrix} 1 & \rho_{yx} \\ \rho_{xy} & 1 \end{bmatrix}$). When (Y, X) follow a bivariate normal distribution, the distributional behavior of B (cf. [2]) entirely depends on n. The distributional properties of the Gini mean differences–based estimators have been discussed by different researchers (cf. David, 1968; Lomnicki, 1952; Nair, 1936; Riaz and Saghir, 2007; Yitzhaki, 2003). From Riaz and Saghir (2007) it follows that G_y is an unbiased estimate of σ_y in the univariate case. They recommended $\overline{G}_y$ mean of G_ys computed from an initial set of stable points for an unbiased estimation of σ_y in the dispersion control charts.

However, the distributional behavior of $G_y^2 G_x^2 - G_{yx} G_{xy}$, $(G_y^2 G_x^2 - G_{yx} G_{xy})^{1/2}$ and hence B is not easy to obtain analytically. Therefore, the Monte Carlo simulation technique has been used to explore the distributional behavior of B. "In practice, simulation methods are often used to evaluate the expectation of a statistic," according to Ross (1990). A detailed discussion regarding the number of simulations required in control chart Monte Carlo simulation studies may be found in Schaffer and Kim (2007). They examined recently published studies to develop recommendations for the minimum number of replications necessary to reproduce the reported results within a specified degree of accuracy. In many cases, only 5,000 replications or fewer were required. In general, the number of replications required to reproduce the target ARL decreased as the shift size increased.

Let b_0, b_1, and B_a, respectively, represent the mean, standard deviation, and ath quantile point (i.e., the point that has a% area below it is completed) of the distribution of B, which entirely depends on n in the case of bivariate normality. The values of b_0, b_1, and B_a have been obtained as function of n using the simulation approach.

To conduct a Monte Carlo experiment, we have generated 10,000 random samples, of a given size n, from the standard bivariate normal distribution without loss of generality. For each sample, we have computed the values of the random variable B

TABLE 1 Table of Coefficients for the $|\underset{\sim}{G}|$-chart

n	b_0	b_1
3	2.540	2.849
4	4.679	3.405
5	6.776	4.072
6	8.851	4.512
7	10.858	4.974
8	12.885	5.358
9	14.815	5.725
10	16.821	5.985
11	18.950	6.357
12	20.878	6.686
13	22.983	7.007
14	24.917	7.308
15	26.865	7.618
16	28.844	7.743
17	31.042	8.206
18	32.935	8.398
19	34.898	8.570
20	37.047	8.843
25	46.964	9.968
30	56.660	10.808
50	97.057	14.197
100	197.341	20.235

followed by its descriptive statistics to obtain b_0, b_1, and B_a. To take care of random variability due to simulation, we have repeated the above procedure 1,000 times. Based on these repeated computations for b_0, b_1, and B_a, we have obtained the mean values of these quantities and their standard errors to report the precision of the results obtained for these quantities (see Tables 1 and 2).

The standard errors for the results of each cell in the aforementioned tables varied between 0.004 and 0.010. Similar results for b_0, b_1, and B_a can easily be obtained for any value of n.

The quantities b_0, b_1, and B_a are needed to determine the control limits and the power of the proposed $|\underset{\sim}{G}|$-chart to detect shifts in the process $|\underset{\sim}{\Sigma}|^{1/2}$.

Design Structure of the Proposed $|\underset{\sim}{G}|$-Chart

Let $\mu_{|\underset{\sim}{G}|^{1/2}}$ and $\sigma_{|\underset{\sim}{G}|^{1/2}}$ denote the mean and the standard deviation of the distribution of the sample

TABLE 2 Table of Quantile Points for the $|\underset{\sim}{G}|$-chart

n	$B_{.001}$	$B_{.005}$	$B_{.01}$	$B_{.050}$	$B_{.100}$	$B_{.200}$	$B_{.800}$	$B_{.900}$	$B_{.950}$	$B_{.990}$	$B_{.995}$	$B_{.999}$
3	0.000	0.000	0.000	0.000	0.000	0.000	4.410	6.375	8.188	12.104	14.305	19.278
4	0.000	0.000	0.000	0.000	1.027	1.849	7.107	9.094	11.134	15.652	17.412	21.044
5	0.000	0.000	0.000	1.692	2.361	3.389	9.758	12.174	14.400	19.510	21.651	27.436
6	0.000	1.231	1.658	2.938	3.762	5.010	12.310	14.892	17.314	22.606	24.703	29.601
7	1.418	2.130	2.618	4.152	5.166	6.629	14.695	17.615	20.258	25.271	27.882	33.086
8	2.081	3.067	3.762	5.538	6.661	8.340	16.915	19.962	22.884	28.305	30.632	36.631
9	3.222	4.183	4.692	6.914	8.226	9.930	19.245	22.605	25.398	31.652	33.788	37.680
10	4.162	5.421	6.136	8.302	9.772	11.688	21.595	24.862	27.752	33.302	35.728	41.954
11	5.269	6.764	7.408	9.796	11.381	13.558	24.125	27.399	30.391	36.475	39.158	45.084
12	6.220	7.652	8.284	11.175	12.950	15.159	26.113	29.820	32.892	39.544	41.873	50.238
13	7.477	8.925	9.777	13.001	14.706	16.936	28.518	32.390	35.551	42.328	45.756	51.836
14	8.019	10.055	10.988	14.164	16.129	18.782	30.765	34.578	38.032	45.066	48.304	53.984
15	9.215	11.485	12.490	15.724	17.663	20.314	32.895	37.082	40.693	47.285	50.755	57.828
16	10.118	12.659	13.732	17.401	19.401	22.225	35.063	39.175	42.447	50.137	52.501	58.772
17	12.452	13.936	15.225	19.073	21.124	24.036	37.678	41.952	45.625	53.463	56.285	62.545
18	12.790	15.274	16.545	20.407	22.770	25.732	39.679	44.083	47.760	55.685	58.716	66.116
19	13.935	16.676	17.981	21.886	24.339	27.572	41.780	46.021	50.107	57.509	60.643	67.891
20	15.623	18.082	19.497	23.799	26.284	29.424	44.225	48.851	52.523	60.557	63.878	69.943
25	22.580	24.882	26.628	31.837	34.830	38.376	55.018	59.892	64.477	73.740	77.247	84.813
30	30.033	32.451	34.017	39.669	43.328	47.588	65.568	70.709	75.584	84.809	88.465	97.046
50	58.195	63.378	66.201	74.642	79.084	84.759	108.858	115.514	121.197	133.191	136.646	145.487
100	140.972	149.265	154.062	165.363	172.214	180.193	213.957	223.752	231.553	248.164	255.027	271.299

statistic $|\underset{\sim}{G}|^{1/2}$, respectively. Applying expectations to [2] gives:

$$\left.\begin{aligned} E(B) &= E(2(n-1)|\underset{\sim}{G}|^{1/2}/|\underset{\sim}{\Sigma}|^{1/2}) \\ &= (2(n-1)/|\underset{\sim}{\Sigma}|^{1/2})E(|\underset{\sim}{G}|^{1/2}) \end{aligned}\right\}. \quad [3]$$

In [3], $E(|\underset{\sim}{G}|^{1/2})$ can be replaced by its estimate $\overline{|\underset{\sim}{G}|}^{1/2}$ (the mean of sample $|\underset{\sim}{G}|^{1/2}$s), using an appropriate number of random samples, from the process under study when the process is in the state of statistical control (cf. Hillier, 1969; Yang and Hillier, 1970)). Thus, from [3] an estimate of $|\underset{\sim}{\Sigma}|^{1/2}$, after rearranging the terms, is given as:

$$|\hat{\underset{\sim}{\Sigma}}|^{1/2} = 2(n-1)\frac{\overline{|\underset{\sim}{G}|}^{1/2}}{b_0}. \quad [4]$$

The expression for $|\hat{\underset{\sim}{\Sigma}}|^{1/2}$ given in [4] is useful for an unbiased estimation of $|\underset{\sim}{\Sigma}|^{1/2}$ using $\overline{|\underset{\sim}{G}|}^{1/2}$ and the coefficient b_0 provided in Table 1 as a function of n.

Also from [3] we have:

$$E(|\underset{\sim}{G}|^{1/2}) = \frac{b_0|\underset{\sim}{\Sigma}|^{1/2}}{2(n-1)}. \quad [5]$$

Replacing the estimate of $|\underset{\sim}{\Sigma}|^{1/2}$ (given in [4]) in [5] and simplification gives:

$$\hat{\mu}_{|\underset{\sim}{G}|^{1/2}} = \overline{|\underset{\sim}{G}|}^{1/2}. \quad [6]$$

Also, taking the variance of B and then simplification gives the expression for σ_B as:

$$\sigma_B = 2(n-1)\sigma_{|\underset{\sim}{G}|^{1/2}}/|\underset{\sim}{\Sigma}|^{1/2}, \quad [7]$$

Rearrangement of (7) yields the following result for $\sigma_{|\underset{\sim}{G}|^{1/2}}$:

$$\sigma_{|\underset{\sim}{G}|^{1/2}} = \frac{b_1|\underset{\sim}{\Sigma}|^{1/2}}{2(n-1)}. \quad [8]$$

Substituting the estimate for $|\underset{\sim}{\Sigma}|^{1/2}$ given in [4] into [8], the estimate for $\sigma_{|\underset{\sim}{G}|^{1/2}}$ is given as:

$$\hat{\sigma}_{|\underset{\sim}{G}|^{1/2}} = \frac{b_1\left(\overline{|\underset{\sim}{G}|}^{1/2}\right)}{b_0}. \quad [9]$$

The expression in [9] is similar to the expression for $\hat{\sigma}_R$ of the R-chart as provided in Alwan (2000).

Parameters of the Proposed $|\underset{\sim}{G}|$-Chart

The central line (*CL*), lower control limit (*LCL*), and upper control limit (*UCL*) are the three parameters of any Shewhart-type control chart. There are two approaches to express these parameters; namely, the probability limit approach and the 3-sigma limit approach. In case of an asymmetric distributional behavior of a relevant estimator, the probability limits approach is preferred. If the distributional behavior of a relevant estimator is nearly symmetric, then the 3-sigma limits approach is a good alternative. In this article we use the probability limits approach because of the asymmetric distributional behavior of the $|\underset{\sim}{G}|$-chart.

Probability Limits Approach

The value $\overline{|\underset{\sim}{G}|}^{1/2}$ corresponds to the *CL* of the proposed $|\underset{\sim}{G}|$-chart. Assuming that the probability of making a Type-I error is less than a specified value, say α, the control limits (which are actually the true probability limits) for the proposed $|\underset{\sim}{G}|$-chart are defined as:

$$\left.\begin{aligned} LCL &= |\underset{\sim}{G}|_l^{1/2} \text{ with } F_n\left(|\underset{\sim}{G}|^{1/2} = |\underset{\sim}{G}|_l^{1/2}\right) \le \alpha_l \\ UCL &= |\underset{\sim}{G}|_u^{1/2} \text{ with } F_n\left(|\underset{\sim}{G}|^{1/2} = |\underset{\sim}{G}|_u^{1/2}\right) \ge 1-\alpha_u \end{aligned}\right\}, \quad [10]$$

where $|\underset{\sim}{G}|_l^{1/2}$ and $|\underset{\sim}{G}|_u^{1/2}$ are the quantile points of the distribution of $|\underset{\sim}{G}|^{1/2}$ below which the areas are α_l and $1-\alpha_u$ respectively, and $\alpha = \alpha_l + \alpha_u$ and $F_n(X=x)$ represent the cumulative distribution function of X at point x, for a given value of n.

Now using [2] and [4] in [10] and simplification gives the following:

$$\left.\begin{aligned} LCL &= |\underset{\sim}{G}|_l^{1/2} = B_l\,\overline{|\underset{\sim}{G}|}^{1/2}/b_0 \text{ with } F_n(B=B_l) \le \alpha_l \\ UCL &= |\underset{\sim}{G}|_u^{1/2} = B_u\,\overline{|\underset{\sim}{G}|}^{1/2}/b_0 \text{ with } F_n(B=B_u) \ge 1-\alpha_u \end{aligned}\right\} \quad [11]$$

Thus, the quantile points of the distribution of B, the average of the sample $|\underset{\sim}{G}|^{1/2}$s (i.e., $|\overline{\underset{\sim}{G}}|^{1/2}$), and the values of b_0 allow setting the true probability limits of the proposed $|\underset{\sim}{G}|$-chart. Similarly, the 3-sigma limits for the proposed $|\underset{\sim}{G}|$-chart may easily be defined using [6] and [9].

Once we have computed the control limits of the proposed $|\underset{\sim}{G}|$-chart for a given significance level by either probability limits approach or the 3-sigma limits approach, the sample statistic $|\underset{\sim}{G}|^{1/2}$ is plotted against the time order of the samples. If all of the sample $|\underset{\sim}{G}|^{1/2}$s lie within the control limits, there is reasonable evidence to conclude that there is no shift in the process $|\underset{\sim}{\Sigma}|^{1/2}$ and that the process is stable. Otherwise, some assignable cause(s) are at work causing a shift in the process $|\underset{\sim}{\Sigma}|^{1/2}$.

To address specifically small and moderate shifts: (i) the runs rules (as discussed by Nelson, 1984) may be supplemented to the basic structure of the proposed $|\underset{\sim}{G}|$-chart of this article, resulting into an increased false alarm rate; (ii) EWMA and CUSUM schemes may be developed based on the statistic $|\underset{\sim}{G}|$ or $|\underset{\sim}{G}|^{1/2}$ (cf. Woodall and Ncube, 1985).

For more than two correlated quality characteristics, the design structure of the proposed $|\underset{\sim}{G}|$-chart may be extended on similar lines. For guidelines regarding more than two correlated quality characteristics of interest, see Gnanadesikan and Gupta (1970) and Khoo and Quah (2004).

PERFORMANCE EVALUATIONS AND COMPARISONS

In this section, the performance of the proposed $|\underset{\sim}{G}|$-chart given earlier is compared with that of the $|\underset{\sim}{S}|$-chart given in Khoo and Quah (2004). The comparisons are made for two different situations; namely, the case of a bivariate normal distribution and the case of non-normal bivariate distributions. For comparison purposes, some selective cases of shifts in the parameter values have been considered and the average run length (ARL) has been computed as the performance measure, as is usually done in quality control literature for comparisons among different methods (cf. Hawkins and Maboudou, 2007; Khoo and Quah, 2004).

The Case of a Bivariate Normal Distribution

The efficiency of the proposed $|\underset{\sim}{G}|$-chart as compared to that of the $|\underset{\sim}{S}|$-chart has been examined here for the case of a bivariate normal distribution, using the ARL as a performance measure. Probability limits of the $|\underset{\sim}{G}|$-chart (cf. [11]) and the $|\underset{\sim}{S}|$-chart have been obtained for different combinations of α and n. In addition, the ARLs for the two charts have been computed. The ARLs for some values of n using $\alpha = 0.005$ are provided here in Tables 3 and 4 for comparisons between the $|\underset{\sim}{G}|$- and the $|\underset{\sim}{S}|$-charts. For the ARL computations, the shifts in $|\underset{\sim}{\Sigma}|^{1/2}$ are considered in terms of $\delta|\underset{\sim}{\Sigma}|^{1/2}$.

One may observe from these tables that (i) for small values of n, the ARLs of the proposed $|\underset{\sim}{G}|$-chart are slightly less than those of the $|\underset{\sim}{S}|$-chart for small values of δ, and the two ARLs almost coincide when δ increases; (ii) for large values of n, the ARLs of the proposed $|\underset{\sim}{G}|$-chart are almost the same as those of the $|\underset{\sim}{S}|$-chart for all the choices of δ.

Thus, the proposed $|\underset{\sim}{G}|$-chart is a competitor to the $|\underset{\sim}{S}|$-chart in case of a bivariate normal parent population for detecting shifts in the process $|\underset{\sim}{\Sigma}|^{1/2}$.

The Case of Non-Normal Bivariate Distributions

Until now we have assumed that the samples are drawn from a normal distribution. In case this is not true, then an option is to employ a control chart appropriately designed for some particular parent distribution. But in practice, we prefer to have control chart structures that are not much affected by the departures from normality. We examine here departures from normality for our proposed $|\underset{\sim}{G}|$-chart and the traditional $|\underset{\sim}{S}|$-chart. To study the effect of non-normality, two situations are considered: one by disturbing the symmetry and the other by disturbing the kurtosis (peak) of the parent distribution. For the case of disturbances in symmetry, we use the bivariate exponential and bivariate chi-square

TABLE 3 ARL Values for the $|\underset{\sim}{G}|$-Chart at $\alpha = 0.005$ for the Bivariate Normal Parent Distribution

δ	$n=5$	$n=10$	$n=20$	$n=50$
1.00	200.011	199.996	200.023	199.992
1.50	29.636	12.919	5.050	1.761
2.00	9.971	3.379	1.555	1.022
2.50	5.106	1.859	1.117	1.001
3.00	3.167	1.383	1.027	1.000
4.00	2.060	1.105	1.001	1.000

TABLE 4 ARL Values for the $|\underset{\sim}{S}|$-Chart at $\alpha = 0.005$ for the Bivariate Normal Parent Distribution

δ	$n=5$	$n=10$	$n=20$	$n=50$
1.00	199.998	199.994	200.031	200.001
1.50	27.362	12.009	5.044	1.760
2.00	8.343	3.217	1.539	1.020
2.50	4.327	1.800	1.115	1.000
3.00	2.900	1.363	1.027	1.000
4.00	1.866	1.101	1.001	1.000

distributions; for the case of disturbance in the kurtosis we use the bivariate t distribution.

To examine the effect of departures from bivariate normality, a bivariate random vector, say $\underline{U}$, is simulated 10,000 times from a bivariate exponential distribution; a bivariate random vector, say $\underline{V}$, is simulated 10,000 times from a bivariate t-distribution; and a bivariate random vector, say $\underline{W}$, is simulated 10,000 times from a bivariate chi-square distribution such that the mean vectors and the variance-covariance matrices of $\underline{U}$, $\underline{V}$, and $\underline{W}$ are the same as that of a comparable random vector from a bivariate normal distribution. Then the calculations are carried out for the charting characteristic B (cf. [2]) based on the simulated random vectors $\underline{U}$, $\underline{V}$ and $\underline{W}$, and the distribution of B is obtained for the three cases under consideration. This is repeated 1,000 times and mean values of the quantile points of the distribution B are obtained. The rejection region for given α is decided, using mean values of the quantile points of the distributions of B derived from $\underline{U}$, $\underline{V}$, and $\underline{W}$, by the probability limits approach and ARLs of the proposed $|\underset{\sim}{G}|$-chart are computed for different shifts in the process $|\underset{\sim}{\Sigma}|^{1/2}$ using the quantile points of Table 2, which are obtained assuming bivariate normality. The same is done, along similar lines, for

TABLE 5 Original and Affected ARLs of the $|\underset{\sim}{G}|$- and $|\underset{\sim}{S}|$-Charts at $\alpha = 0.005$ for $n = 5$

δ	G-chis5	S-chis5	G-chis20	S-chis20	G-Expo	S-Expo	G-t5	S-t5	G-t20	S-t20
1.00	200.020	200.033	199.998	199.990	200.013	199.999	199.989	200.041	200.015	200.020
1.50	63.291	122.768	41.017	49.428	270.270	1710.267	147.059	244.165	43.478	44.281
2.00	15.552	23.949	11.429	12.585	40.783	161.364	26.596	40.273	11.891	11.637
2.50	6.930	9.422	5.429	5.850	14.493	39.545	10.428	13.914	5.618	5.522
3.00	4.182	5.278	3.503	3.653	7.539	16.236	5.794	7.150	3.596	3.495
4.00	2.364	2.741	2.118	2.161	3.501	5.785	2.975	3.345	2.150	2.100

TABLE 6 Original and Affected ARLs of the $|\underset{\sim}{G}|$- and $|\underset{\sim}{S}|$-Charts at $\alpha = 0.005$ for $n = 10$

δ	G-chis5	S-chis5	G-chis20	S-chis20	G-Expo	S-Expo	G-t5	S-t5	G-t20	S-t20
1.00	200.016	200.007	200.001	199.999	199.993	200.027	200.023	200.014	200.008	199.992
1.50	25.126	77.232	18.692	24.738	86.957	1846.092	73.421	663.600	16.795	18.576
2.00	4.895	9.559	4.163	4.860	9.372	70.218	8.741	36.843	3.934	4.119
2.50	2.354	3.503	2.132	2.300	3.517	13.102	3.387	8.472	2.043	2.082
3.00	1.600	2.069	1.505	1.582	2.090	5.077	2.026	3.747	1.474	1.488
4.00	1.169	1.306	1.142	1.166	1.314	2.019	1.297	1.730	1.132	1.138

TABLE 7 Original and Affected ARLs of the $|\underset{\sim}{G}|$- and $|\underset{\sim}{S}|$-Charts at $\alpha = 0.005$ for $n = 20$

δ	G-chis5	S-chis5	G-chis20	S-chis20	G-Expo	S-Expo	G-t5	S-t5	G-t20	S-t20
1.00	200.018	200.023	200.015	199.994	200.021	200.016	200.013	200.022	199.990	200.008
1.50	7.257	45.109	5.857	7.351	8.772	456.192	11.236	426.376	5.631	6.475
2.00	1.791	3.974	1.640	1.778	1.927	13.010	2.142	12.548	1.622	1.691
2.50	1.176	1.634	1.139	1.171	1.214	2.958	1.271	2.902	1.134	1.151
3.00	1.004	1.184	1.033	1.043	1.055	1.569	1.072	1.553	1.003	1.037
4.00	1.004	1.019	1.003	1.003	1.005	1.075	1.006	1.073	1.002	1.003

TABLE 8 Original and Affected ARLs of the $|\underset{\sim}{G}|$- and $|\underset{\sim}{S}|$-Charts at $\alpha = 0.005$ for $n = 50$

δ	G-chis5	S-chis5	G-chis20	S-chis20	G-Expo	S-Expo	G-t5	S-t5	G-t20	S-t20
1.00	200.012	199.997	200.010	200.021	199.997	199.995	200.023	200.011	200.010	200.009
1.50	1.605	7.781	1.804	2.417	1.674	66.253	2.138	102.276	1.843	2.129
2.00	1.002	1.240	1.028	1.047	1.022	2.123	1.005	2.432	1.003	1.035
2.50	1.001	1.011	1.001	1.001	1.001	1.085	1.002	1.112	1.001	1.001
3.00	1.000	1.000	1.000	1.000	1.001	1.005	1.000	1.008	1.000	1.000
4.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

the $|\underset{\sim}{S}|$-chart using the quantile points of a chi-square distribution with $2n-4$ degrees of freedom for comparison purposes.

Tables 5 through 8 provide the ARLs of the $|\underset{\sim}{G}|$- and $|\underset{\sim}{S}|$-charts when the samples are generated from the comparable bivariate exponential, t and chi-square distributions, for some selective values of n using $\alpha = 0.005$.

In the above tables, G(S)-Chi5, G(S)-Chi20, G(S)-t5, G(S)-t20, and G(S)-Expo refer to the ARLs of the $|\underset{\sim}{G}|$ ($|\underset{\sim}{S}|$)-chart when the parent distributions are bivariate chi-square with 5 (respectively 20) degrees of freedom, bivariate t with 5 (respectively 20) degrees of freedom, and bivariate exponential, respectively.

From Tables 5 through 8 we may conclude that the ARLs of the proposed $|\underset{\sim}{G}|$-chart are less affected by non-normality compared to those of the $|\underset{\sim}{S}|$-chart. A similar behavior is observed for the other values of n. Thus, the proposed $|\underset{\sim}{G}|$-chart provides a reasonably robust design structure that can be used even if the behavior of the correlated quality characteristics (Y, X) depart from normality.

An Interesting Phenomenon for the Exponential Distribution

It has been observed that in the case of exponential distribution, the values of ARL increase instead of decrease for small shifts (e.g., $\delta = 1.5$) for both $|\underset{\sim}{G}|$- and $|\underset{\sim}{S}|$-charts. This increase in the ARL values is larger for the $|\underset{\sim}{S}|$-chart compared to that of the $|\underset{\sim}{G}|$-chart, especially for small values of n, as is obvious from Tables 5 through 7. A similar type of observation for the exponential distribution was also made by Vermaat and Does (2006) in their study regarding a semi-Bayesian method for Shewhart control charts.

Moreover, another aspect of the proposed $|\underset{\sim}{G}|$-chart is covered in Riaz (2008), which shows that this proposed chart is not unduly affected by the presence of contaminations (outliers and special causes).

CONCLUSIONS

This article shows that the proposed $|\underset{\sim}{G}|$-chart is a satisfactory competitor to the $|\underset{\sim}{S}|$-chart, in case of a normal parent population. Moreover, the design structure of the proposed $|\underset{\sim}{G}|$-chart shows more robust behavior compared to that of the $|\underset{\sim}{S}|$-chart against departures from normality.

ACKNOWLEDGEMENTS

The authors thank the referees and the editor for their constructive criticism, which has improved the article substantially.

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